Optimal Advertising When Envisioning a Product-Harm Crisis

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Abstract

How should forward-looking managers plan advertising if they envision a product-harm crisis in the future? To address this question, we propose a dynamic model of brand advertising in which, at each instant, a non-zero probability exists for the occurrence of a crisis event, which damages the brand’s baseline sales and may enhance or erode marketing effectiveness when the crisis occurs. Because managers don’t know when the crisis will occur, its random time of occurrence induces a stochastic control problem, which we solve analytically in closed-form. More importantly, the envisioning of a possible crisis alters managers’ rate of time preference: anticipation enhances impatience. That is, forward-looking managers discount the present—even when the crisis has not occurred—more than they would in the absence of crisis. Building on this insight, we then derive the optimal feedback advertising strategies and assess the effects of crisis likelihood and damage rate. We discover the crossover interaction: the optimal pre-crisis advertising decreases, but the post-crisis advertising increases as the crisis likelihood (or damage rate) increases. In addition, we develop a new continuous-time estimation method to estimate sales dynamics and feedback strategies simultaneously using discrete-time data. Applying it to market data from the Ford Explorer’s rollover recall, we furnish evidence to support the proposed model. We detect compensatory effects in parametric shift: ad effectiveness increases, but carryover effect decreases (or vice versa). We also characterize the crisis occurrence distribution that shows Ford Explorer should anticipate a crisis in 2.1 years and within 6.3 years at the 95% confidence level. Finally, we find a remarkable correspondence between the observed and optimal advertising decisions.

Keywords: Product Harm Crisis; Optimal Advertising; Stochastic Optimal Control; Random Stopping Time Problem; Kalman Filter; Ford Explorer Rollover.

One-line Abstract: Crises erode profit —so conserve now, and recover later!
1. Introduction

Companies face a realistic chance of encountering product-harm crisis (e.g., Toyota’s unintended acceleration, BP’s oil spill, Ford-Firestone tire recall). Due to the crisis of unintended acceleration in the winter 2010, Toyota Camry recalled 8 million cars, lost 16% sales, and dropped 12 billion dollars in market value. BP’s oil spill could have resulted in a bankruptcy. Ford-Firestone crisis resulted in claims worth 3.5 billion dollars and ended their century old relationship. Thus, product harm crisis arrives unannounced and damages a brand’s sales and profitability.

To help managers respond to crisis events, consulting companies render advice such as tell the truth to relevant groups; contain the crisis; repair the damage; and re-build the brand (Augustine 1995, O’Donnell 2009). To understand customers’ response, experimental studies investigate the effects of hypothetical crises on consumer expectations (Dawar and Pillutla 2000), commitment to the brand (Ahluwalia et al. 2000), brand loyalty (Stockmeyer 1996), perceived locus of the problem (Griffin, Babin and Attaway 1991), or prior corporate social responsibility (Klein and Dawar 2004). To gauge the extent of damage to baseline sales and marketing effectiveness, empirical studies estimate dynamic sales models after the crisis occurs. For example, the study by van Heerde, Helsen and Dekimpe’s (2007) show that peanut butter crisis in Australia damaged the brand’s baseline sales and attenuated the effectiveness of its marketing mix activities. Thus, the literature studies ex-post consequences of crises using the three types of studies: managerial, experimental, and empirical. But managers should expect the crisis to occur unexpectedly; so how should they make ex ante advertising decisions differently if they envision a crisis in the future?

The extant literature, to best of our knowledge, contains no normative study that sheds light on this question. Consequently, we do not know the effects of incorporating the risk of crisis in advertising decisions. Specifically, should forward-looking managers increase —or
decrease—pre-crisis advertising if crisis likelihood increases? If managers anticipate brand sales to plummet, how should they optimally set advertising budget *ex ante* (i.e., before the crisis)? How would they estimate the crisis likelihood using sales-advertising data? How much should they spend if they aspire to an impressive comeback (e.g., Tylenol contamination)? Should competing brands exploit the misfortunes of the focal brand?

These questions are simple to ask, but hard to answer. To see this point, consider the decision dilemma: should managers increase or decrease the pre-crisis advertising if they envision a crisis in the future? One argument suggests that pre-crisis advertising would insulate the brand from negative publicity when it occurs, which requires managers to increase ad spending (Cleeren, Dekimpe and Helsen 2008). On the other hand, investments in pre-crisis advertising would be partly wasted due to the likely damage to baseline sales, which requires them to decrease ad spending. Both the arguments—the insulation effect and investment effect—are valid, and so we cannot resolve the opposing recommendations without formal analysis.

Hence, we formulate a dynamic advertising model in which, at each instant, a non-zero probability exists for the occurrence of a crisis event, which hurts the brand’s sales and may erode or enhance marketing effectiveness when the crisis occurs. The random occurrence time induces a stochastic control problem, which we solve analytically to derive the optimal feedback advertising strategies in closed-form. Applying comparative statics analysis, we deduce the effects of crisis likelihood and damage rate. We next extend the basic model to incorporate recovery dynamics and competition. In addition, we investigate the role of size-dependent insulation effect (i.e., larger brands are less vulnerable to sales loss), investments in quality (i.e., improving quality to reduce crisis likelihood), and optimal pricing strategy. Finally, we apply the basic model to Ford Explorer’s rollover crisis, which damaged the baseline sales by 35% immediately after the crisis (compared to 16% for
Toyota’s recent crisis).

The above theoretical and empirical analyses make three broad contributions. First, the analytical model introduces a novel feature that the crisis event occurs at a random time in the future. Previous dynamic models incorporated uncertainty via continuous Brownian motion (i.e., the Wiener process), which represents the effects of several small shocks whose net impact on average is zero. In contrast, crisis causes uncertainty via a discrete event whose net impact is not zero. We furnish a general framework to study uncertainty from rare catastrophic events, which is large and discrete rather than small and continuous. The resulting dynamic model augments the class of control-theoretic models in marketing (e.g., see Feichtinger, Hartl and Sethi 1994; Sethi and Thompson 2000) by introducing a random stopping problem (see Section 3), which facilitates the study of long-term optimal decisions in the presence of rare but catastrophic events (Boukas, Haurie and Michel 1990, Haurie and Moresino 2006).

Second, besides this modeling contribution, we offer seven new propositions on managing advertising in the presence of a potential product-harm crisis. Specifically, the mere act of envisioning a crisis alters managers’ time preference: crisis likelihood enhances impatience. That is, forward-looking managers discount the present—even in the pre-crisis regime when the crisis has not occurred—more than they would in the absence of crisis. In resolving the dilemma described earlier, we discover the crossover interaction: managers should decrease the pre-crisis advertising and increase the post-crisis advertising as crisis likelihood (or damage rate) increases. Conserve spending now; defend sales later. We elucidate the intuition in Section 4.

Third, this paper is the first one to develop a method for estimating both the sales dynamics and feedback strategies simultaneously in continuous-time using discrete-time data (see section 5.2). Empirical results establish the presence of compensatory effects in
parametric shift: ad effectiveness increases, but carryover effect decreases (or vice versa) after the crisis. This finding complements and augments the previous results on uniform attenuation reported by van Heerde et al. (2007). Moreover, we show how to estimate the crisis likelihood and characterize the distribution of random crisis occurrence. Results suggest that Explorer should expect a crisis in about 2 years and within 6 years at the 95% confidence level. Finally, by comparing the observed and optimal ad spending, we find a remarkable correspondence after the crisis: actual advertising is 1.7% above the optimal spends for Ford Explorer; 4.4% for Jeep Cherokee; and 1.9% for Toyota 4Runner. Thus, the proposed model comports with the managers’ actions.

The rest of the paper proceeds as follows. Section 2 develops the dynamic brand sales model with a random jump due to the crisis event. Section 3 evaluates the long-term profit impact of the crisis event, and Section 4 derives the optimal feedback advertising strategies and the effects of crisis likelihood and sales damage. Section 5 presents the market data from Ford Explorer rollover recall and lends empirical support for the proposed model. Section 6 discusses several related issues (e.g., size-dependent insulation effect, investments in quality, pricing decisions), and section 7 concludes by identifying avenues for future research.

2. Model Development

2.1. Brand Sales Dynamics

Let \( u(t) \) denote a firm’s advertising spending at time \( t \) to build the brand’s sales \( S(t) \). For example, Ford Explorer’s advertising \( u(t) \) exceeds one million dollars every week. The resulting growth in brand sales can be described by the nonlinear extension of the Vidale-Wolfe model formulated by Sethi (1983),

\[
\frac{dS}{dt} = \beta_j \sqrt{u_j(t)} \sqrt{M(t) - S(t)} - \delta_j S(t),
\]  

(1)
where $\beta_j$ represents ad effectiveness, $\mu_j(t)$ captures the diminishing returns to advertising, $(M - S)$ is the untapped market with $M$ as the market size, $\delta_j$ is the decay rate, and the index $j = 1$ and 2 denotes the pre- and post-crisis regimes, respectively.

Equation (1) states that the ad spending $u(t)$ results in sales growth $dS$ during the time interval $dt$ by capturing some portion of the untapped market; in the absence of advertising, sales decline at the rate proportional to the prevailing sales level. Previous advertising studies have used this model in empirical and analytical research (e.g. Feichtinger, Hartl and Sethi 1994, Erickson 2003, Naik, Prasad, and Sethi 2008, Erickson 2009b).

The main difference from previous models is the probabilistic switch in the regimes due to a crisis. Equation (1) specifies the evolution of the one continuous state variable $S(t)$ and one discrete state variable ($j = 1$ or $j = 2$) that identifies the regime in which the brand operates (i.e., pre- or post-crisis). The index $j$ indicates that all model parameters differ before and after the crisis. In the study on crisis impact, van Heerde et al. (2007) find that ad effectiveness and carryover effect drop below their corresponding pre-crisis levels. Although van Heerde et al. (2007) assume they know when the crisis occurred in their data (i.e., deterministic regime switch), the crisis occurs at an unknown date *ex ante* when planning for the next year’s advertising budget. To this end, we describe the probabilistic evolution of this binary state variable.

2.2. Probabilistic Regime Switch Due to a Crisis

Because crisis strikes at an unknown instant in the future, the timing of crisis occurrence is random. Let $\{I(t): t \geq 0\}$ denote the crisis occurrence process in which the crisis may occur at any instant $t$ with probability $\chi \in (0,1)$. Further, let $T$ be the random date at which the crisis actually occurs. Then, the horizon splits into two sub-intervals: the pre-crisis regime ($j = 1$) where $t \in [0,T)$ before the crisis, and the post-crisis regime ($j = 2$) where
$t \in [T, \infty)$ after the crisis. Consequently, the process \{\Gamma(t): t \geq 0\} is a jump process with the jump rate defined by

$$\lim_{dt \to 0} \frac{P[\Gamma(t + dt) = 2 | \Gamma(t) = 1]}{dt} = \chi \quad \text{and} \quad \lim_{dt \to 0} \frac{P[\Gamma(t + dt) = 1 | \Gamma(t) = 2]}{dt} = 0.$$  \hspace{1cm} (2)

To complete the model formulation, we specify the initial conditions for the pre- and post-crisis regimes. At $t = 0$, the initial sales level is

$$S(0) = S_{10}. \hspace{1cm} (3)$$

Based on van Heerde et al. (2007), the baseline sales drop after the product-harm crisis, and we capture this “damage” as follows:

$$S(T^+) = (1 - \phi)S(T^-), \hspace{1cm} (4)$$

where $S(T^+)$ and $S(T^-)$ represent the sales levels just after and right before the crisis event, respectively. The fraction $\phi$ is the damage rate: the larger the damage rate, the sharper the drop in the baseline sales, which hurts the long-term profit.

### 3. Long-term Profit Impact of a Crisis

Here we formulate the advertising decision problem, evaluate the long-term profit as a function of crisis likelihood, and discover how crisis alters decision-making.

#### 3.1 Advertising Decisions in the Presence of a Crisis

In continuous-time framework, the net present value of brand advertising is given by

$$J(u) = \int_0^\infty e^{-\rho t} \pi(S(t),u(t))dt, \quad \text{where } \rho \text{ denotes the discount rate, } \pi(S,u) = mS - u \text{ is the profit function, } m \text{ is the unit margin (e.g., see the review articles by Sethi 1977, Little 1979, Feichtinger et al. 1994). When brand managers envision a crisis event at the random date } T \text{ in the future, they compute the profits from the pre- and post-crisis regimes. Let } (m_1, m_2) \text{ denote the margins in the two regimes, respectively. Although the brand’s margin can be the same in}$$
both the regimes, we relax them in the following analysis. For example, \( m_2 < m_1 \) can be thought of as an increase in the variable cost to improve quality (e.g., Tylenol designed tamper-proof packaging). Then the net present value from each regime is

\[
J_1(u_1) = \int_0^T e^{-\rho t} \pi(S(t), u_1(t)) dt \quad \text{and} \\
J_2(u_2) = \int_T^\infty e^{-\rho t} \pi(S(t), u_2(t)) dt .
\]

In Equation (5a), \( J_1 \) is the total discounted profit from \( t = [0, T) \), and it accrues at time \( t = 0 \). Likewise, Equation (5b) yields the total discounted profit from \( t = [T, \infty) \) and the resulting \( J_2 \) becomes available at time \( t = T \).

Note that \( T \) is a random variable because the manager does not know \textit{ex ante} at \( t = 0 \) the exact date when the crisis would occur. The random date \( T \) takes any value on the time line \([0, \infty)\). Consequently, the net present value \( J_1 \) is also a random variable; \( J_1 \) is small if \( T \) occurs early and large if \( T \) occurs later. Hence, by taking the expectation over all possible values of \( T \), we compute the expected net present value of profit from both the regimes,

\[
J(u_1, u_2) = E[J_1(u_1) + e^{-\rho T} J_2(u_2)] ,
\]

where the expectation \( E[\cdot] \) is taken with respect to the crisis occurrence process in (2). The discount factor \( e^{-\rho T} \) appears in (6) because \( J_1 \) accrues at \( t = 0 \) and \( J_2 \) at \( t = T \), requiring the discounting of \( J_2 \) back to \( t = 0 \) to add the two long-term profits.

The resulting expected long-term profit in (6) depends on the choice of advertising feedback strategies \((u_1(S), u_2(S))\). The forward-looking managers take into account the inter-temporal trade-off in spending resources now versus later. In addition, they incorporate the probability \( \chi \) with which a crisis might strike. We denote the optimal advertising strategies \((u_1^*, u_2^*)\), which are feedback solutions to the optimization problem:
(u_1^*(S), u_2^*(S)) = ArgMax J(u_1(S(t)), u_2(S(t)))

subject to the sales dynamics in (1), the random crisis occurrence process in (2), and the initial values in pre- and post-crisis periods via equations (3) and (4), respectively.

Equation (7) represents a stochastic control problem because it involves a discrete random event of crisis occurrence (due to equation (2)). Previous marketing studies formulated stochastic control problems using the Wiener process (e.g., Rao 1986, Nguyen 1997, Raman and Chatterjee 1995, Raman and Naik 2004, Prasad and Sethi 2008). The uncertainty structure of the Wiener process represents many small shocks of uncertain magnitude at every instant whose net impact on average is zero. However, the product harm crisis induces a different uncertainty structure qualitatively. Specifically, it involves timing uncertainty, damages the baseline sales, and attenuates or amplifies marketing effectiveness. In other words, we face a large shock at an uncertain time whose net impact on average is not zero. To solve this discrete event stochastic control problem, we evaluate the expectation in (6).

3.2 Random Stopping Problem

When the crisis occurs, the pre-crisis regime ends. The timing of its occurrence is random. Let f(t) and F(t), respectively, denote the probability density and cumulative distribution functions of the crisis occurrence process in (2). Because the crisis can occur at any instant t given that it has not occurred thus far in the interval [0, t), the crisis hazard rate \( h(t) = \chi \). Then the distribution function is given by \( F(t) = 1 - \exp(-\int_0^t h(s)ds) \), and the density function by \( f(t) = h(t) \times (1 - F(t)) = \chi \exp(-\int_0^t \chi ds) \). Using this density function \( f(t) \), we evaluate the expectation in (6) as follows:
\[ J(u_1, u_2) = E\left[ \int_0^t e^{-\rho s} \pi(s) ds + e^{-\rho s} J_2 \right] \]

\[ = \int_0^t \left[ e^{-\rho s} \pi(s) ds + e^{-\rho s} J_2 \right] (\chi e^{-\rho t}) dt \] (8)

\[ = I_1 + \chi \int_0^\infty e^{-\rho t + \chi t} J_2 dt \]

In the Appendix, we prove that

\[ I_1 = \int_0^\infty e^{-\rho t + \chi t} \pi(t) dt. \] (9)

Finally, substituting (9) into (8) and simplifying, we obtain the total long-term profit from both the regimes:

\[ J(u_1, u_2, \chi) = \int_0^\infty e^{-\rho t + \chi t} \left\{ \pi(S(t), u_1(t)) + \chi J_2(u_2(t)) \right\} dt. \] (10)

Equation (10) is the long-term profit as a function of advertising decisions and the crisis likelihood. It reveals an insight into how the presence of a crisis event alters decision-making.

To see this insight, compare equations (5) and (10) and observe that the discount rate increases from \( \rho \) to \( \rho + \chi \). Thus, the envisioning of a crisis alters the manager’s rate of time preference. Specifically, forward-looking managers discount the present — even in the pre-crisis regime when the crisis has not occurred — more than they would in the absence of crisis. Thus, they become more impatient as they envision crisis scenarios, leading us to the result:

**Proposition 1.** Crisis likelihood enhances impatience.

### 4. Effects of Crisis on Optimal Advertising

We first formulate and solve the value functions to derive the optimal feedback advertising strategies and then deduce the effects of crisis likelihood and damage rate.

#### 4.1 Value Functions in Pre- and Post-Crisis Regimes
The goal of forward-looking managers is to maximize the long-term profit by selecting the best ad spending at each instant $t$ in both the pre- and post-crisis regimes. Formally, they should consider all possible strategies $(u_1(S), u_2(S))$ and select the best ones $(u_1^*(S), u_2^*(S))$ that maximize the long-term profit equation (10). When they use the optimal advertising strategies, the largest value of the long-term profit in (10) is given by

$$V(S_{10}, S(T^+)) = \max_{u_1, u_2} J(u_1(t), u_2(t), \chi)$$

starting from any initial sales $S_{10}$ before the crisis and initial $S(T^+)$ just after the crisis. In equation (11), forward-looking managers incorporate the profit from the second regime, $J_2(\cdot)$, when deciding their actions in the pre-crisis regime. Consequently, the maximum value of $J_2(\cdot)$ is necessary to maximize (11). Hence, we apply the backward induction principle and start with the second regime first.

In the second regime, let $W(S(T^+)) = \max_{u_2} J_2(u_2(t))$ represent the value function, which yields the maximum long-term profit, starting from the new baseline sales $S(T^+)$ after the crisis. Because the time left after the crisis from $[T, \infty)$ is the same as $[0, \infty)$, we translate the origin and solve the control problem:

$$W(S(T^+)) = \max_{u_2} \int_0^\infty e^{-\rho t} \{m_2 S(t) - u_2\} dt$$

subject to

$$\frac{dS}{dt} = \beta_2 \sqrt{u_2} \sqrt{M(t) - S(t)} - \delta_2 S(t), \quad S(T^+) = (1 - \phi) S(T^-)$$

In the Appendix, we solve for the value function and derive the optimal post-crisis feedback advertising strategy.

Substituting the resulting value function $W(S(T^+))$ from (12) in (11), we then solve the control problem in the first regime, which is given by
\[ V(S_{10}) = \max_{u_1} \int_{0}^{\infty} e^{-(\rho + \delta_s) t} \left\{ m_1 S(t) - u_1 + \chi W((1 - \phi)S(t)) \right\} dt \]

subject to

\[ \frac{dS}{dt} = \beta_1 \sqrt{u_1} \left( M(t) - S(t) - \delta_1 S(t) \right), \quad S(0) = S_{10} \]

In the Appendix, we also derive the optimal pre-crisis feedback advertising strategy. Next, we state both the optimal strategies in Proposition 2.

### 4.2 Optimal Advertising

Let \( \lambda_1 \) and \( \lambda_2 \), respectively, denote the long-term profitability in the pre- and post-crisis regimes. We interpret \( \lambda_j \) as the marginal long-term profit due to incremental sales in regime \( j \) (e.g., Kamien and Schwarz 2003, p. 136). In the Appendix, we solve the Hamilton–Jacobi-Bellman equation associated with equations (12) and (13) to derive the optimal feedback advertising strategies, which we present in Proposition 2:

**Proposition 2.** The optimal pre- and post-crisis feedback advertising strategies, respectively, are given by

\[ u_1^*(S(t)) = (M(t) - S(t))(0.5\beta_1 \lambda_1)^2 \]

and

\[ u_2^*(S(t)) = (M(t) - S(t))(0.5\beta_2 \lambda_2)^2, \]

where

\[ \lambda_1 = \frac{-2(\rho + \delta_s + \chi) + 2\sqrt{(\rho + \delta_s + \chi)^2 + (m_1 + \chi(1 - \phi)\lambda_2)^2}}{\beta_1^2} \]

and

\[ \lambda_2 = \frac{-2(\rho + \delta_s) + 2\sqrt{(\rho + \delta_s)^2 + m_2 \beta_2^2}}{\beta_2^2}. \]

**Proof.** See the Appendix.

The above feedback advertising possesses the desirable property of *sub-game perfectness*. That is, the optimal advertising spending is optimal not only when sales evolve along the optimal state trajectory of the starting game, but also when sales depart from the optimal trajectory in *any* sub-game, which may (i) occur probabilistically at any time due to a crisis, (ii) induce any possible sales drop, and (iii) alter any model parameter(s). This property is also known as Markov perfectness (Zaccour 2008, p. 90) or strong time consistency (Basar and Olsder 1999).
In the absence of crisis ($\chi = 0$), the optimal ad spending $u^*_1(S) = u^*_2(S)$ (as it should be), and it depends on the ad effectiveness $\beta$, the margin $m$, and the decay and discount rates ($\delta, \rho$). This result informs managers to spend more on advertising when ad effectiveness or brand profitability increases (for example, due to a new creative that is more effective or a new technology that reduces variable costs). Furthermore, managers should spend more when sales are low and less when sales are high; that is, spend intensively when the untapped market is large.

In the presence of a potential crisis, the forward-looking managers should alter their pre- and post-crisis ad spending based on the following two factors: the likelihood $\chi$ of crisis occurrence (which enhances impatience) and the damage rate $\phi$ during the crisis (which leads to the loss in baseline sales). To gain further insights and contribute new results to marketing science, we conduct comparative statics analysis of optimal strategies with respect to the above factors.

### 4.3 Crisis Likelihood Effects

Before we present the results, we re-state the decision dilemma: should managers increase or decrease advertising as crisis likelihood increases? One argument—the insulation effect—favors an increase in pre-crisis advertising because advertising builds brand equity that “buffers” the potential damage after the crisis (Cleeren, Dekimpe and Helsen 2008, p. 262). A counter-argument—the investment effect—recommends a decrease in pre-crisis advertising to offset the loss in profit they expect due to the crisis (e.g., lost baseline sales).

Both arguments are valid. Indeed, the long-term profit embodies the insulation and investment effects of crisis. Specifically, in equation (10), the crisis likelihood $\chi$ appears in both the exponent term and the multiplier of the post-crisis profit $J_2$. The exponent term, as stated in Proposition 1, enhances managers’ impatience, which reduces the level of effort
they exert in the pre-crisis regime and thus decreases the pre-crisis ad spending as $\chi$ increases. In contrast, the multiplier term increases the total profit, thereby increasing the pre-crisis ad spending as $\chi$ increases. Because the two opposing forces coexist, the extant literature cannot shed light in general (i.e., for every feasible parameter value) whether optimal advertising should increase or decrease as $\chi$ increases? Hence, to resolve this dilemma, we apply comparative statics analysis to obtain the new findings:

**Proposition 3** (*Crisis Likelihood Effects.*)

*As the crisis likelihood $\chi$ increases, the optimal pre-crisis advertising decreases, whereas the optimal post-crisis advertising increases.*

**Proof.** See the Appendix.

Why should managers reduce advertising when they simply consider the possibility of a crisis? Because they become impatient as $\chi$ increases as per Proposition 1 and so they reduce the brand-building effort. It is remarkable that even though crisis affects the post-crisis regime, it influences pre-crisis decisions. Why? Because the principle of backward induction requires managers to look forward and then reason backwards to make dynamically optimal decisions.

**[Insert Figure 1 here]**

Figure 1 illustrates the trajectories of sales and advertising in the presence of low and high crisis likelihood. First, it shows that pre-crisis ad spending is lower for higher $\chi$ at every $t < T$. Second, at the crisis time $T$, advertising increases to recover the lost sales. Third, we observe a larger post-crisis advertising for higher $\chi$ at every $t > T$.¹ This result follows from the mediating effects of $\chi$ via $u^*_1 \rightarrow S_1(t) \rightarrow S_1(T^-) \rightarrow S_2(T^+) \rightarrow S_2(t) \rightarrow u^*_2$ reported in Figure 2. The direct effect of $\chi$ (or $\phi$) on $u^*_2$ is zero.

**[Insert Figure 2 here]**

¹ We acknowledge the contributions of the Area Editor in this mediation analysis.
Thus we discover the *crossover* interaction: $u_1^*$ decreases but $u_2^*$ increases when managers envision a crisis. They should conserve resources now to recover sales later.

Finally, we emphasize that constant hazard rate provides a benchmark to learn how managers should adapt their advertising in the presence of negative or positive duration dependence. Specifically, Proposition 3 informs that optimal advertising should be reduced if the hazard rate increases (positive dependence) and enhanced if it decreases (negative dependence). Explicit time-varying hazard analysis (not reported here) for negative and positive duration dependence comport with these predictions.

### 4.4 Damage Effects

**Proposition 4** (*Damage Effects.*)

*As the damage rate $\phi$ increases, the optimal pre-crisis advertising decreases, whereas the optimal post-crisis advertising increases.*

**Proof.** See the Appendix.

The intuition for this result is as follows. A crisis damages the baseline sales. When managers anticipate this damage, they recognize the potential loss in profit in the post-crisis regime. Because they maximize the sum of profits from both the regimes, a reduction in pre-crisis ad spending offsets the anticipated loss in the post-crisis regime. In other words, they plan ahead by reducing the pre-crisis ad spending to compensate future losses. Hence, pre-crisis ad spending decreases as the damage rate increases.

The post-crisis sales are affected by the combination of two adverse forces. First, the reduction in pre-crisis advertising suppresses the sales trajectory, leading to the lower sales $S_1(T^-)$. Second, the damage rate $\phi$ further reduces $S_1(T^+)$ via Equation (4). Hence, post-crisis advertising increases to compensate for the double whammy: brand sales starts lower at $t = T$ and drops even further due to $\phi$. See Figure 1.

### 4.5 Recovery
How would advertising strategies differ when a brand manager envisions recovery after the crisis? To investigate this question, let the crisis last for a random duration so that the brand recovers at time $\tau$. To account for this “come back” to the pre-crisis state, we augment the process $I(t)$ as follows:

$$
\lim_{dt \to 0} P(\Gamma(t + dt) = 2 | \Gamma(t) = 1) = \chi \quad \text{and} \quad \lim_{dt \to 0} P(\Gamma(t + dt) = 1 | \Gamma(t) = 2) = \omega,
$$

where $\omega \in (0,1)$ is the intensity of the recovery process. The recovery is characterized as normal ($\theta = 0$) or impressive ($\theta > 0$), which bumps up the baseline sales as follows:

$$S(\tau^+) = (1 + \theta)S(\tau^-),$$

(15)

To find the optimal strategy $u_2$ in the crisis regime, the brand manager faces the dynamic optimization problem,

$$W(S(T^+)) = \max_{u_2} \int_0^\infty e^{-(\rho + \omega)t} \left[ m_2 S(t) - u_2 + \omega V((1 + \theta)S(t)) \right] dt$$

subject to $dS/dt = \beta_2 \sqrt{u_2} M(t) - S - \delta_2 S$, and $S(T^+) = (1 - \phi)S(T^-)$.

In the pre-crisis regime, to obtain $u_1$, it faces the dynamic optimization problem,

$$V(S(\tau^+)) = \max_{u_1} \int_0^\infty e^{-(\rho + \omega)t} \left[ m_1 S(t) - u_1 + \omega W((1 - \phi)S(t)) \right] dt$$

subject to $dS/dt = \beta_1 \sqrt{u_1} M(t) - S - \delta_1 S$, and $S(\tau^+) = (1 + \theta)S(\tau^-)$.

(17)

We solve the optimization problems in (16) and (17) and present the results in the next proposition.

**Proposition 5.** In the presence of recovery dynamics, the optimal feedback advertising strategies are given by

$$u_1^*(S(t)) = (M(t) - S(t))(0.5 \beta_1 \lambda_1)^2 \quad \text{and} \quad u_2^*(S(t)) = (M(t) - S(t))(0.5 \beta_2 \lambda_2)^2,$$

where $\lambda_1$ and $\lambda_2$ are the solutions to the algebraic equations:

$$\lambda_1 = \frac{-2(\rho + \delta_1 + \chi) + 2\sqrt{((\rho + \delta_1 + \chi)^2 + (m_1 + \chi(1 - \phi)\lambda_2)^2)}}{\beta_1^2},$$

and

$$\lambda_2 = \frac{-2(\rho + \delta_2 + \omega) + 2\sqrt{((\rho + \delta_2 + \omega)^2 + (m_2 + \omega(1 + \theta)\lambda_1)^2)}}{\beta_2^2}.$$
Proof. See the Appendix.

Because $\lambda_2$ is an increasing function of $\theta$, managers should increase advertising during recovery. The proposed model assumes a monopolistic setup to understand how managers should advertise when they envision a crisis. But how should other competing brands react to the focal brand’s crisis? Should they exploit its misfortune? To answer these questions, we apply differential game theory as in previous marketing research (e.g., Chintagunta and Vilcassim 1992, Chintagunta and Vilcassim 1994, Jørgensen and Zaccour 2003, Naik, Prasad, and Sethi 2005, Bass et al. 2005, Erickson 2009a, Erickson 2009b, Fruchter and Mantrala 2010).

4.6 Competition

Let $A$ denote the focal brand anticipating the crisis, $B$ be the competing brand, and $i = \{A, B\}$ index the two brands. Each brand’s sales is affected by not only its own advertising $u_j(S_A, S_B)$ and own sales, but also the market penetration of the competitor’s brand as follows:

$$dS_i/dt = \beta_i u_i \sqrt{M - S_i} - S_i - \delta_i S_i.$$ \hfill (18)

If crisis occurs, brand $A$ would lose its sales and increase the available market potential to $B$, thus fostering advertising competition. Each brand envisions this scenario and determines simultaneously its optimal advertising strategy to maximize the long-term profit. Suppressing $t$ for notational clarity, the following proposition furnishes the optimal advertising strategies in Nash equilibrium.

Proposition 6. The Nash equilibrium feedback advertising strategies by the focal brand before and after the crisis, respectively, are given by

$$u_{A1}^*(S_A, S_B) = (M - S_A - S_B)(0.5 \beta_{A1}\lambda_{A1})^2$$ and $$u_{A2}^*(S_A, S_B) = (M - S_A - S_B)(0.5 \beta_{A2}\lambda_{A2})^2.$$  

Similarly, the Nash equilibrium feedback advertising strategies by the competing brand before and after the crisis, respectively, are given by
\begin{align*}
u_{B_1}^*(S_A, S_B) &= (M - S_A - S_B)(0.5\beta_{B_1}\mu_{B_1})^2 \quad \text{and} \quad u_{B_2}^*(S_A, S_B) = (M - S_A - S_B)(0.5\beta_{B_2}\mu_{B_2})^2.
\end{align*}

The parameters $(\lambda_{Aj}, \lambda_{Bj}, \mu_{Aj}, \mu_{Bj})$ are solutions to the algebraic equations given in the Appendix.

**Proof.** See the Appendix.

Comparing this proposition with Proposition 2, we learn that the focal brand changes its advertising strategy in the presence of competing brands. Furthermore, $B$ increases its advertising when it anticipates $A$’s sales loss. In other words, the competing brand exploits the misfortunes of the focal brand. The above duopoly dynamic game can be generalized to oligopoly markets with any number of brands (see section 5 for a triopoly market).

In sum, the normative analysis reveals that a crisis induces impatience (see Proposition 1); that the optimal advertising before (after) the crisis decreases (increases) when envisioning a crisis (see Proposition 3); that the optimal pre-crisis advertising decreases, but the post-crisis advertising increases as the anticipated loss in baseline sales increases (see Proposition 4); that the optimal advertising increases during recovery (see Proposition 5); that the competing brands increase advertising to exploit the misfortunes of the focal brand (see Proposition 6).

We next develop a new method to estimate continuous-time models using discrete-time data that jointly specifies the sales dynamics and feedback strategies. We apply it to Ford Explorer’s rollover recall data, obtain parameter estimates, and then compute the crisis likelihood and damage rate.

5. **Empirical Analyses**

5.1 Ford Explorer Recall

On August 7\textsuperscript{th}, 2000, Ford recalled the Explorer sports utility vehicle because it consistently displayed tendencies to rollover (Govindaraj, Jaggi and Lin 2004). This recall spurred over 200 lawsuits (New York Times 2002) and severed a century-old relationship.

In this study, we examine the pre-recall period October 1996 through August 7, 2000, the recall period from August 7, 2000 to December 10, 2000, and the post-recall period from December 10, 2000 until June 30, 2002. A leading data supplier in the automotive industry provided data on pre- and post-crisis weekly sales and prices for major SUV brands sold in California. TNS Inc. furnished the advertising expenditures for the corresponding weeks. Before the recall, Explorer’s average market share was 14.3%, which plummeted to 4.1% during the recall, and then increased to 5.5% after the recall. Figure 3 displays the weekly sales (see Panel A) and advertising (see Panel B) before and after the recall. To account for competitive effects, we included the Jeep Cherokee and Toyota 4-Runner brands, which constitute the top three brands in the SUV category. Table 1 presents the descriptive statistics on sales and advertising of the three brands and the SUV market size over time.

[Insert Table 1 and Figure 3 about here]

5.2 Continuous-time Estimation

Let \( i \in \{A, B, C\} \) denote Ford Explorer, Jeep Cherokee, and Toyota 4-Runner, respectively. Suppressing the regime subscript \( j = \{1, 2\} \) for notational clarity (although each regime has its own parameter vector), we generalize (18) to a triopoly market to obtain the sales dynamics:

\[
\dot{S}_A = \beta_A \sqrt{u_A} \sqrt{M - S_A - S_B - S_C} - \delta_A S_A,
\]

\[
\dot{S}_B = \beta_B \sqrt{u_B} \sqrt{M - S_A - S_B - S_C} - \delta_B S_B,
\]

\[
\dot{S}_C = \beta_C \sqrt{u_C} \sqrt{M - S_A - S_B - S_C} - \delta_C S_C,
\]

where the dot denotes time-derivate (i.e., \( \dot{x} = \frac{dx}{dt} \)). As in Proposition 6, we derive the three-brand feedback advertising strategies:
\begin{equation}
\begin{aligned}
u_A &= (M - S_A - S_B - S_C)(0.5 \beta_A \lambda_A)^2 \\
u_B &= (M - S_A - S_B - S_C)(0.5 \beta_B \mu_B)^2 \\
u_C &= (M - S_A - S_B - S_C)(0.5 \beta_C \gamma_C)^2 \\
\end{aligned}
\end{equation}

Then we substitute (20) into (19) to obtain the continuous-time sales dynamics:

\begin{equation}
\begin{bmatrix}
\dot{S}_A \\
\dot{S}_B \\
\dot{S}_C
\end{bmatrix} = 
\begin{bmatrix}
-\delta_A - 0.5 \beta_A^2 \lambda_A & 0.5 \beta_A^2 \lambda_A & 0.5 \beta_A^2 \lambda_A \\
-0.5 \beta_B^2 \mu_B & -\delta_B - 0.5 \beta_B^2 \mu_B & 0.5 \beta_B^2 \mu_B \\
-0.5 \beta_C^2 \gamma_C & 0.5 \beta_C^2 \gamma_C & -\delta_C - 0.5 \beta_C^2 \gamma_C
\end{bmatrix}
\begin{bmatrix}
S_A \\
S_B \\
S_C
\end{bmatrix} +
\begin{bmatrix}
0.5 \beta_A^2 \lambda_A M \\
0.5 \beta_B^2 \mu_B M \\
0.5 \beta_C^2 \gamma_C M
\end{bmatrix}
\end{equation}

Equation (21) is a vector differential equation system, \( \dot{S} = RS + r \), where \( \dot{S} = (\dot{S}_A, \dot{S}_B, \dot{S}_C)' \).

The time \( t \) in (21) is continuously differentiable on any interval, \( t < t < \tilde{t} \). However, observed data arrive at discrete points in time (e.g., weeks). That is, in the data series, the time parameter \( t_k \) is not continuously differentiable; rather it takes discrete values in the integer set \( \{1, 2, 3, \ldots, N\} \). To estimate continuous-time models using discrete-time data, Rao (1986) integrates the univariate sales dynamics and applies ordinary least squares to the resulting equation to obtain parameter estimates. Naik, Raman, and Winer (2005) extend this approach to multivariate sales dynamics with coupled differential equations and apply the Kalman filter to estimate model parameters. Below we further extend their approach to estimate not only the multivariate sales dynamics, but also the feedback advertising strategies simultaneously.

To this end, by applying the method of integrating factor, we integrate (21) over the interval \( [t_{k-1}, t_k] \),

\[
S_k - S_{k-1} = \int_{t_{k-1}}^{t_k} dS = \int_{t_{k-1}}^{t_k} (dS/dt) dt = \int_{t_{k-1}}^{t_k} (\dot{S}) dt
\]

and we obtain the matrix equation,

\begin{equation}
S_k = Z_k S_{k-1} + d_k,
\end{equation}

21
where $Z_1 = \text{Exp}(R)$ is the transition matrix, $\text{Exp}()$ is the matrix exponentiation function, and $d_1 = (\text{Exp}(R) - I)A^{-1}r$ with $I$ being a 3 x 3 identity matrix. We observe that, unlike Rao (1986) and Naik et al. (2005), we do not assume constancy of advertising during $[t_{k-1}, t_k)$ because we incorporate advertising dynamics via the feedback strategies in (20). Also, note that the matrix exponentiation is not the same as the exponentiation of the elements of the matrix in general. The decay rate $\delta_i \in \mathbb{R}^+$ and is not restricted to the unit interval as in discrete-time estimation. Consequently, we compute the usual carryover effects from the diagonal elements $Z_{ii}$. Because $Z_{11}$ depends on $(\delta_A, \beta_A, \lambda_A)$, and likewise for the other diagonal elements, we learn that the carryover effects depends not only on the sales decay, but also on ad effectiveness and brand profitability.

Next, we concatenate (22) with the feedback advertising strategies in (20) to obtain the state-space model:

$$
\begin{bmatrix}
S_k \\
u_k
\end{bmatrix} =
\begin{bmatrix}
Z_1 & 0_{3 \times 3} \\
Z_2 & 0_{3 \times 3}
\end{bmatrix}
\begin{bmatrix}
S_{k-1} \\
u_{k-1}
\end{bmatrix} +
\begin{bmatrix}
d_1 \\
d_2
\end{bmatrix} +
\begin{bmatrix}
\varepsilon_k \\
\nu_k
\end{bmatrix},
$$

where $u_k = (u_A, u_B, u_C)'$, $\varepsilon_k \sim N(0, \sigma^2)\quad$ and $\nu_k \sim N(0, \sigma^2)$ are sales and advertising transition noise, respectively, and

$$
Z_2 =
\begin{bmatrix}
-0.25\beta_A^2\lambda_A^2 & -0.25\beta_A^2\lambda_A^2 & -0.25\beta_A^2\lambda_A^2 \\
-0.25\beta_B^2\mu_B^2 & -0.25\beta_B^2\mu_B^2 & -0.25\beta_B^2\mu_B^2 \\
-0.25\beta_C^2\gamma_C^2 & -0.25\beta_C^2\gamma_C^2 & -0.25\beta_C^2\gamma_C^2
\end{bmatrix},
$$

and $d_2 = \begin{bmatrix}
0.25\beta_A^2\lambda_A^3\mu_{k-1} \\
0.25\beta_B^2\mu_B^3\mu_{k-1} \\
0.25\beta_C^2\gamma_C^3\mu_{k-1}
\end{bmatrix}$.

Denoting $\alpha_k = (S_k, u_k)'$ and expressing (23) as $\alpha_k = Z\alpha_{k-1} + d_k + \xi_k$, we link it to the observation equation, $Y_k = \alpha_k + \xi_k$, where $Y_k$ contains the observed sales and advertising data over time. We then apply the Kalman filter (see Naik, Mantrala, Sawyer 1998 for details) to compute the evolving moments of $\alpha_k$ for $t_k = 1, \ldots, N$. Using those moments, we compute the log-likelihood of observing the joint sales sequence $\Omega_N = (Y_1, Y_2, \ldots, Y_N)'$, which is given by
\[
L(\Theta; \Omega_k) = \sum_{k=1}^{N} \ln(p(Y_k | \Omega_{k-1})), \quad (24)
\]

where \( p(\cdot | \cdot) \) is the conditional density of \( Y_t \) given the sequence of sales up to the last period, \( \Omega_{k-1} \). The \( K \times 1 \) vector \( \Theta \) contains the model parameters from each regime \( j \), the error variances, and the initial mean of \( \alpha_0 \). We maximize (24) with respect to \( \Theta \) to obtain the maximum-likelihood estimates:

\[
\hat{\theta} = \text{ArgMax} \; L(\Theta). \quad (25)
\]

Then, we conduct robust statistical inference (White 1982). Specifically, if the model was correctly specified, we would obtain the standard errors from the square-root of the diagonal elements of the inverse of the matrix:

\[
Q = \left[ -\frac{\partial^2 L(\Theta)}{\partial \theta \partial \theta'} \right]_{\hat{\theta}}, \quad (26)
\]

where \( Q \) equals the negative of the Hessian of \( L(\Theta) \) evaluated at the estimated values \( \hat{\theta} \).

However, because the model is likely misspecified, we aim to make inferences robust to unknown misspecification errors. To obtain the robust standard errors, we apply Huber-White correction known as the sandwich estimator (see White 1982):

\[
\text{Var}(\hat{\theta}) = Q^{-1} P Q^{-1}, \quad (27)
\]

where \( P \) is a \( K \times K \) matrix of the gradients of the log-likelihood function; that is, \( P = G' G \), and \( G \) is \( N \times K \) matrix obtained by stacking the \( 1 \times K \) vector of the gradient of the log-likelihood function for each of the \( N \) observations. For correctly specified models, \( Q = P \) and so both the equations (26) and (27) yield exactly the same standard errors (as they should); otherwise, we use the robust standard errors given by the square-root of the diagonal elements of (27). Thus, the robust standard errors safeguard us from the unknown forms of specification errors (White 1982).

[Insert Table 2 about here]
Finally, we compare the proposed model with the specification proposed by Bass, Krishnamoorthy, Prasad, and Sethi (2005) that incorporates the cross-effects of competing brand’s advertising on own sales. Because their model does not contain the effects of crisis likelihood and damage rate, first, we extend their model as well to incorporate those effects across both regimes. Second, we derive the feedback strategies for their extended model across both the pre- and post-crisis regimes. Third, by applying the above method, we estimate this extended model and present the resulting information criteria in Table 2. In Panel A, we observe that the extended Bass et al (2005) model yields higher scores, indicating inferior predictive performance. Furthermore, both AIC and BIC metrics indicate the same outcome, which enhances our confidence in the findings. Hence, we retain the proposed model based on model selection theory (e.g., Burnham and Anderson 2002, p. 70).

Fourth, we investigate whether the structure of sales dynamics changes across regimes. Specifically, we consider the proposed model (M1) and the Bass et al. (2005) model (M2) to switch or to continue across regimes. In other words, we estimate four pairs of dynamic models across the two regimes: (M1, M1), (M2, M2), (M1, M2), and (M2, M1). Moreover, each model has its own regime- and brand-specific parameters. Panel B of Table 2 reports the resulting information criteria. We find that the proposed model (M1, M1) attains the minimum score, indicating better predictive performance than that for the other three combinations. These results hold across both the AIC and BIC metrics, enhancing our confidence in the results. Hence, market data lends support to the proposed model.

[Insert Table 3 about here]

In Table 3, we display the estimated parameters and their robust t-values for the proposed model. First, across all the three brands, all the parameter estimates have expected signs and are significant at the 95% confidence level. Second, for both the American brands,

\[\text{[Insert Table 3 about here]}\]

2 We thank an anonymous reviewer for further extending this empirical analysis.
ad effectiveness increases and carryover effect decreases after the crisis. In contrast for Toyota 4-Runner, ad effectiveness decreases after the crisis, whereas its carryover effect increases. The last two findings suggest the presence of compensatory effect: ad effectiveness and carryover effect move in opposite directions.

To understand these empirical results, consider the data from the National Highway Transportation Safety Administration (NHTSA) reported by Mayne et al. (2001). Specifically, Ford’s share of all vehicles recalled was 32.1% (33.3%) in 2000 (1999) and that for Jeep was 28.2% (29.7%). Moreover, both Ford and Jeep were the two largest SUV brands. Indeed, “both brands … were perceived as similar by consumers in terms of quality, safety, performance, and aesthetics” (Vakratsas and Ma 2009, p. 29). Hence, according to Feldman-Lynch theory, the common attribute of “Americanness” becomes diagnostic and accessible to consumers, making the main competitor (Jeep) of the scandalized brand (Explorer) “…considered guilty by association” (Roehm and Tybout 2006, p. 366). Consequently, the enhanced media scrutiny focuses attention to American brands at the expense of the Japanese manufacturer, accounting for the ad effectiveness findings.

Furthermore, NHTSA data report that Toyota’s share of all vehicles recalled was as low as 0.03% (3.1%) in 2000 (1999). This greater (lower) reliability of Toyota (American brands) results in higher (lower) customer retention rates, accounting for the carryover effects.

In sum, our empirical results differ qualitatively from the uniform attenuation of model parameters documented by van Heerde et al. (2007). Thus, we augment and complement the extant empirical literature. Specifically, we augment their findings by establishing the presence of compensatory shift in parameters. We complement the sparse extant literature by studying a different product category (cars versus peanut butter), where consumers’ behavior differ because of infrequent purchase incidence and high price-to-income ratio.
5.3 Crisis Estimation

To characterize the crisis occurrence distribution, we first observe from the data that the damage rates are $\hat{\phi}_A = 0.346$ (Ford), $\hat{\phi}_B = 0.153$ (Jeep), $\hat{\phi}_C = -0.123$ (Toyota). Then, using the estimated parameters $(\hat{\beta}, \hat{\delta}, \hat{\lambda}_A, \hat{\lambda}_B, \hat{\lambda}_C)^T$ for $i = \{A, B, C\}$, we compute the crisis likelihood as follows. The feedback strategies in (20) yield the analytical expressions for the coefficients of the value functions. Specifically, in each regime, we get three equations for each of the three brands (i.e., $3 \times 3 = 9$ per regime). Finally, for a given discount rate, we can solve these eighteen equations to find the unknown coefficients, which include the crisis likelihood. Accordingly, for $\rho = 6\%$ per annum, we compute $\hat{\chi} = 0.476$ per annum. Based on section 3.2, the density function of the random crisis timing $T$ is $f(t) = \chi \exp(-\chi t)$. Hence, the expected crisis time for Ford Explorer is $E[T] = \chi^{-1} \approx 2.1$ years. In addition, the percentiles of $T$ are given by $F^{-1}(p; \chi) = -\frac{\ln(1-p)}{\chi}$, and so the 95th percentile $T_{0.95} = (-\ln(1-0.95))/\chi = 6.3$ years. Applying this procedure to different discount rates, we find a robust range for the expected crisis timing. Specifically, we doubled the discount rate to $\rho = 12\%$ per annum to get $E[T] = 1/0.403 \approx 2.5$ years; we tripled it to $\rho = 18\%$ per annum to get $E[T] = 1/0.331 \approx 3.0$ years. Thus, Ford Explorer should expect a crisis in 2 to 3 years.

[Insert Table 4 about here]

We close this section by furnishing the confidence interval for the pre- and post-crisis optimal advertising. To this end, we apply Krinsky and Robb’s (1986) approach to obtain confidence intervals. Specifically, we randomly draw 10,000 points from the asymptotic distribution of the estimated parameters and then evaluate (20) for the optimal pre- and post-crisis advertising. The resulting 2.5th percentile and 97.5th percentile covers the 95% confidence interval. Table 3 reports the confidence intervals. We observe that average ad
spending for each brand in each regime (see Table 1) lies well within their corresponding intervals. Although over-spending exists before the crisis, actual advertising after the crisis is 1.7% above the optimal spends for Ford Explorer; 4.4% for Jeep Cherokee; and 1.9% for Toyota 4Runner. Thus, the optimal decisions comport with the managers’ actions.

6. Discussions

We discuss size-dependent insulation effect, investment in quality to reduce the crisis likelihood, and pricing strategy.

6.1. Size-dependent Insulation Effect

In the proposed model, we assumed the sales damage is independent of brand’s size. But suppose larger brands suffer a smaller loss in baseline sales because of their brand equity. To account for this size-dependent insulation effect, we extend the damage function \( \phi(S) = \phi_0 + \phi_1/S \) (with \( S \geq 1 \)). Then we obtain the result:

**Proposition 7.** The optimal pre- and post-crisis feedback advertising strategies are

\[
\nu_1^*(S(t)) = (M(t) - S(t))(0.5 \beta_1 \lambda_1)^2 \quad \text{and} \quad u_2^*(S(t)) = (M(t) - S(t))(0.5 \beta_2 \lambda_2)^2,
\]

respectively, where

\[
\lambda_1 = \frac{-2(\rho + \delta_1 + \chi) + 2\sqrt{(\rho + \delta_1 + \chi)^2 + (m_1 + \chi(1 - \phi_0)\lambda_2)^2 \beta_2^2}}{\beta_1^2},
\]

\[
\lambda_2 = \frac{-2(\rho + \delta_2) + 2\sqrt{(\rho + \delta_2)^2 + m_2 \beta_2^2}}{\beta_2^2}.
\]

**Proof.** See the Appendix.

Although Proposition 7 appears similar to Proposition 2, it yields two new insights. First, optimal ad spending under size-dependent insulation effect exceeds that under size-independent insulation effect. Thus, intuitively, when insulation effect depends on the brand’s size, the brand manager seeks to grow sales by spending more on advertising, thereby

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3 We thank an anonymous reviewer for this suggestion.
reducing the damage rate. Second, Proposition 7 reveals that the optimal pre-crisis ad spending does not depend on $\phi_1$, which is the coefficient for size-dependent effect. Why? Because brands have incentive to grow and reduce the damage rate; consequently, they invest in advertising to attain the largest profitable size and the smallest damage rate (which is $\phi_0$).

6.2. Investments in Quality

If product-harm crisis is brewing, should managers invest in improving the product quality? Chen, Ganesan, Liu (2009) find that the firm’s market value reduces substantially more under the proactive strategy (i.e., recall the defective products before a crisis occurs) than under the reactive strategy (i.e., wait for the crisis to occur before initiating the recall). In other words, investors impose larger financial penalty for the proactive strategy because they infer that the impending crisis must be severe enough for managers to recall the product sold. So investment to improve the quality of products sold may not be financially prudent.

However, should managers invest in quality improvement for future production runs? That is, managers can reduce the crisis probability by investing resources to enhance product quality. Let \{v, \overline{v}\} denote low or high investment in quality enhancement, which results in a high or low crisis likelihood \{\overline{\chi}, \chi\}, respectively. Then, in the Appendix, we derive the new result that a brand should increase both ad spending and quality investments to lower the crisis likelihood only if its sales exceed a certain threshold; else, it reduces both the ad spending and quality investment and bears a higher risk of crisis.

6.3. Optimal Pricing Strategy

To set optimal prices, managers can utilize the framework developed by Bass et al. (2005, p. 560), which can be extended for the crisis case. To illustrate briefly, the HJB equations of for the two competing brands $A$ and $B$ in the pre-crisis regime are

$$(\rho + \chi) V_i(S_i, S_g) = \max_{\alpha_i, \beta_i} \left\{ (p_i - c_i)(1 - \alpha_{i} p_i + b_i p_{i+1}) S_i - u_i + \chi V_i((1 - \phi_3) S_i, (1 - \phi_2) S_g) \right\}$$

$$_{i \in \{A, B\}} \sum_{\delta_i} \frac{\partial W_i}{\partial S_i} \left( \beta_i \sqrt{\mu_i} \sqrt{M - S_i - S_g - \delta_i S_i} \right)$$
where \( \left(1 - a_{i1}p_{i1} + b_{i1}p_{3-i,1}\right) \) captures the sales shrinkage due to price competition (see Footnote 2 in Bass et al. 2005). As the Appendix shows, the optimal prices in regime \( j \) are given by

\[
p_{adj} = \frac{b_{adj} + a_{adj}(2 + 2a_{adj}c_{adj} + b_{adj}c_{adj})}{4a_{adj}a_{adj} - b_{adj}b_{adj}} \quad \text{and} \quad p_{bij} = \frac{b_{adj} + a_{adj}(2 + 2a_{adj}c_{adj} + b_{adj}c_{adj})}{4a_{adj}a_{adj} - b_{adj}b_{adj}}.
\]

The feedback advertising strategies are similar to those in Proposition 6, but the associated algebraic equations (see (A46)-(A49) and (A53)-(A56) in the Appendix) depend on \( a_{i1} \) and \( b_{i1} \). Analyzing this equilibrium, we learn that intensified price competition reduces optimal advertising because the long-term profitability diminishes.

7. Conclusions

A French adage “gérer, c’est prévoir” enlightens us that “managing is envisioning.” Hence, brand managers should envision the unexpected. They can estimate the crisis likelihood and damage rate as illustrated via the analyses of Ford Explorer’s sales-advertising data. Specifically, the expected crisis time ranges between 2 to 3 years; a crisis occurs within 6 years at the 95% confidence level; it damages the baseline sales by 35%.

Nonetheless, the next crisis will strike at a random time, and hence managers should envision the risk *ex ante* and incorporate it in their decision-making. One way to manage this risk is to reduce the pre-crisis ad spending and increase the post-crisis ad spending to recover lost sales. In competitive markets, non-focal brands should increase advertising to exploit the sales lost by the focal brand. Finally, managers can reduce the crisis likelihood by investing in quality of future products.

We contribute not only new substantive and empirical results, but also methodological and modeling innovations. Specifically, a novel substantive insight is that brand managers should change their pre-crisis advertising decisions even though a crisis
affects the post-crisis parameters. Why? Because forward-looking managers anticipate a reduction in baseline sales and/or long-term profitability after the crisis. This anticipation enhances impatience, which reduces pre-crisis advertising and thereby induces a double whammy. First, the sales level at the crisis time is lower because of the reduced pre-crisis advertising; additionally, the brand loses customers when the crisis occurs. To recover this sales loss due to double adverse effects, the optimal response is to increase advertising after the crisis. In a nutshell, managers should reduce ad spending before the crisis, but advertise more after it occurs when needed.

A novel empirical insight is that the parameters change in a compensatory manner: ad effectiveness increases, while carryover effect decreases (or vice versa). Due to media scrutiny, customers focus attention on focal or similarly categorized brands, which enhances ad effectiveness; but it makes salient their lower reliability, hurting customer retention and carryover effects. This parametric shift differs qualitatively from the uniform attenuation documented by van Heerde et al. (2007). Collectively, our findings complement the sparse empirical literature on the impact of product-harm crisis.

A novel methodological contribution is the development of a method to estimate continuous-time models using discrete-time data that jointly specifies the sales dynamics and feedback strategies. We learn that the decay rate is not restricted to the unit interval as in discrete-time estimation, and that the carryover effect depends on not only the decay rate, but also ad effectiveness and brand profitability.

Finally, a novel modeling innovation is the nature of uncertainty from rare catastrophic events, which is large and discrete rather than small and continuous as formulated previously (e.g., Raman and Chatterjee 1995, Raman and Naik 2004, Prasad and Sethi 2008). The resulting random stopping problem and its solution approach in section 3 are new to marketing science. We encourage researchers to adapt this framework to solve
other open marketing problems (e.g., optimal decisions in the presence of competitive entry
or technological innovation).

We close by identifying two important topics for further investigation. We restricted
the scope of our normative analyses to product-harm crisis because of our empirical
application. However, the nature of the random timing of crisis and its impact on profitability
(see section 3) permit a broader interpretation of what constitutes a crisis. For example,
financial crisis or recessions or wars are also catastrophic events beyond management’s
control. We encourage future researchers to use the proposed framework and appropriate data
to investigate the marketing implications of such general crises. Second, in this study, the
value function after the crisis is $W_A = 3276 S_A - 59 S_B - 435 S_C + 2.5 \times 10^7$, which shows that
the cash flows from profits remain positive for the prevailing own and competitors’ sales.
Although bankruptcy does not arise in our case, we encourage future research to investigate
this phenomenon using control-theoretic models (see Sethi 1997) and thereby improve the
theory and practice of crisis management.
Table 1. Descriptive Statistics

<table>
<thead>
<tr>
<th>Variables</th>
<th>Pre-Crisis</th>
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<th>Post-Crisis</th>
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<tbody>
<tr>
<td></td>
<td>Average</td>
<td>Deviation</td>
<td>Average</td>
<td>Deviation</td>
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<tr>
<td><strong>Sales (units/week)</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Ford Explorer</td>
<td>727</td>
<td>190</td>
<td>428</td>
<td>200</td>
</tr>
<tr>
<td>Jeep Cherokee</td>
<td>514</td>
<td>119</td>
<td>380</td>
<td>137</td>
</tr>
<tr>
<td>Toyota 4Runner</td>
<td>470</td>
<td>113</td>
<td>338</td>
<td>103</td>
</tr>
<tr>
<td><strong>Advertising ($1,000/week)</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Ford Explorer</td>
<td>883</td>
<td>811</td>
<td>1,485</td>
<td>1,985</td>
</tr>
<tr>
<td>Jeep Cherokee</td>
<td>1,744</td>
<td>971</td>
<td>1,056</td>
<td>576</td>
</tr>
<tr>
<td>Toyota 4Runner</td>
<td>237</td>
<td>98</td>
<td>312</td>
<td>506</td>
</tr>
<tr>
<td><strong>Market Size (units/week)</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>5,317</td>
<td>1,534</td>
<td>8,028</td>
<td>1,195</td>
</tr>
</tbody>
</table>
Table 2. Models Comparison

**Panel A**

<table>
<thead>
<tr>
<th>Model</th>
<th>Pre-Crisis AIC (BIC)</th>
<th>Post-Crisis AIC (BIC)</th>
</tr>
</thead>
<tbody>
<tr>
<td>M1 = Proposed Model</td>
<td>4948.6 (4593.0)</td>
<td>2396.4 (2237.1)</td>
</tr>
<tr>
<td>M2 = Bass et al. 2005</td>
<td>6029.3 (5705.4)</td>
<td>3022.3 (2890.5)</td>
</tr>
</tbody>
</table>

**Panel B**

<table>
<thead>
<tr>
<th>Models Pre- and Post-Crisis</th>
<th>AIC (BIC)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(M1, M1)</td>
<td>7345.1 (6830.1)</td>
</tr>
<tr>
<td>(M2, M2)</td>
<td>9051.6 (8595.9)</td>
</tr>
<tr>
<td>(M1, M2)</td>
<td>7970.9 (7483.5)</td>
</tr>
<tr>
<td>(M2, M1)</td>
<td>8425.7 (7942.5)</td>
</tr>
<tr>
<td>Parameters</td>
<td>Pre-Crisis Estimates (Robust t-values)</td>
</tr>
<tr>
<td>------------</td>
<td>----------------------------------------</td>
</tr>
<tr>
<td><strong>Ford Explorer</strong></td>
<td></td>
</tr>
<tr>
<td>Advertising Effect, $\beta_A$</td>
<td>0.0023 (4.51)</td>
</tr>
<tr>
<td>Sales Decay, $\delta_A$</td>
<td>0.1888 (5.09)</td>
</tr>
<tr>
<td>Carryover Effect, $Z_{11}$</td>
<td>0.8043</td>
</tr>
<tr>
<td>Value Function Coefficient, $\lambda_A$</td>
<td>12606</td>
</tr>
<tr>
<td><strong>Jeep Cherokee</strong></td>
<td></td>
</tr>
<tr>
<td>Advertising Effect, $\beta_B$</td>
<td>0.0084 (9.06)</td>
</tr>
<tr>
<td>Sales Decay, $\delta_B$</td>
<td>1.3640 (8.61)</td>
</tr>
<tr>
<td>Carryover Effect, $Z_{22}$</td>
<td>0.2241</td>
</tr>
<tr>
<td>Value Function Coefficient, $\mu_B$</td>
<td>5005</td>
</tr>
<tr>
<td><strong>Toyota 4Runner</strong></td>
<td></td>
</tr>
<tr>
<td>Advertising Effect, $\beta_C$</td>
<td>0.0284 (34.25)</td>
</tr>
<tr>
<td>Sales Decay, $\delta_C$</td>
<td>1.7052 (47.87)</td>
</tr>
<tr>
<td>Carryover Effect, $Z_{33}$</td>
<td>0.1688</td>
</tr>
<tr>
<td>Value Function Coefficient, $\gamma_C$</td>
<td>524</td>
</tr>
</tbody>
</table>
Table 4: Observed versus Optimal Advertising

<table>
<thead>
<tr>
<th></th>
<th>Pre-Crisis ($000)</th>
<th>Post-Crisis ($000)</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Ford Explorer</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>95% Confidence Interval for Optimal Advertising</td>
<td>(315, 1386)</td>
<td>(1098, 1903)</td>
</tr>
<tr>
<td>Optimal Advertising</td>
<td>717</td>
<td>1460</td>
</tr>
<tr>
<td>(Observed-Optimal)/Optimal Advertising</td>
<td>23.2%</td>
<td>1.7%</td>
</tr>
<tr>
<td><strong>Jeep Cherokee</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>95% Confidence Interval for Optimal Advertising</td>
<td>(663, 2912)</td>
<td>(760, 1317)</td>
</tr>
<tr>
<td>Optimal Advertising</td>
<td>1507</td>
<td>1011</td>
</tr>
<tr>
<td>(Observed-Optimal)/Optimal Advertising</td>
<td>15.7%</td>
<td>4.4%</td>
</tr>
<tr>
<td><strong>Toyota 4Runner</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>95% Confidence Interval for Optimal Advertising</td>
<td>(84, 368)</td>
<td>(230, 399)</td>
</tr>
<tr>
<td>Optimal Advertising</td>
<td>190</td>
<td>306</td>
</tr>
<tr>
<td>(Observed-Optimal)/Optimal Advertising</td>
<td>24.7%</td>
<td>1.9%</td>
</tr>
</tbody>
</table>
Figure 1. Crisis Likelihood Effects on Advertising and Sales Trajectories
Figure 2. Mediation Effects

![Diagram of mediation effects with variables \( \chi \), \( u_1^* \), \( S_1(t) \), \( S_1(T^-) \), \( S_2(T^+) \), \( S_2(t) \), and \( u_2^* \).]
Figure 3. Ford Explorer Sales and Brand Advertising

Panel A. Weekly Brand Sales Volume of Ford Explorer

Panel B. Weekly Brand Advertising of Ford Explorer
APPENDIX

Proof of Equation (9). To evaluate $I_1$ in Equation (8), we apply the integration by parts,

$$\int UdV = UV - \int VdU,$$

where we define $U(t) = \int_0^t e^{-\rho s} \pi(s) ds$ and $V(t) = -e^{-\rho t}$ so that

$$dU = e^{-\rho t} \pi(t) dt \quad \text{and} \quad dV = \chi e^{-\rho t} dt.$$  

Making the appropriate substitutions, we obtain

$$I_1 = \left[ \int_0^t e^{-\rho s} \pi(s) ds \right] \chi e^{-\rho t} dt$$

$$= (-U(t)e^{-\rho t})\bigg|_0^\infty - \int_0^\infty (-e^{-\rho t})(e^{-\rho t} \pi(t) dt)$$

$$= 0 + \int_0^\infty e^{-(\rho+\chi) t} \pi(t) dt,$$

where the first term vanishes because $\lim_{t \to \infty} e^{-\rho t} = 0$ and $U(0) = \int_0^\infty e^{-\rho s} \pi(s) ds = 0$.

We next prove the Propositions 2 through 7 and offer derivations to support the other results.

Proof of Proposition 2 - Optimal Advertising Strategies

We suppress $t$ for notational clarity. Applying backward induction, we first solve the problem in the post-crisis regime. The Hamilton-Jacobi-Bellman (HJB) equation is

$$\rho W(S) = \max_{u_2} \{ m_2 S - u_2 + \frac{\partial W}{\partial S} (\beta S \sqrt{M-S} - \delta S) \}. \quad (A1)$$

By differentiating the parenthetical expression in (A1) with respect to (wrt) $u_2$, we obtain the first order condition (FOC):

$$-1 + \frac{\beta S \sqrt{M-S}}{2 \sqrt{u_2}} \frac{\partial W}{\partial S} = 0 \quad \Rightarrow \quad u_2^*(S) = (M-S)(0.5 \beta S \frac{\partial W}{\partial S})^2. \quad (A2)$$

We substitute (A2) in (A1) to obtain an ordinary differential equation (ODE) in $W(S)$. To solve this ODE, we apply the method of underdetermined coefficients. Specifically, we first conjecture

$$W(S) = \lambda_2 S + \lambda_{02} \quad (A3)$$

and then identify the coefficients as

$$\lambda_2 = \frac{-2(\rho + \delta)}{\beta \lambda_2} + 2\sqrt{(\rho + \delta)^2 + m_2 \beta^2}, \quad \text{and} \quad (A4)$$

$$\lambda_{02} = \frac{M (\beta \lambda_2)^2}{4 \rho}. \quad (A5)$$

Hence, the post-crisis feedback strategy is $u_2^*(S(t)) = (M(t) - S(t))(0.5 \beta_2 \lambda_2)^2$, proving one-half of Proposition 2.
In the pre-crisis regime, the HJB equation is
\[(\rho + \chi)V(S) = \max_{u_1} \{m_1 S - u_1 + \chi W((1 - \phi)S) + \frac{\partial V}{\partial S}(\beta_i \sqrt{u_1 \sqrt{M - S - \delta_i S}})\}. \tag{A6}\]
We obtain the FOC wrt \(u_1\),
\[-1 + \frac{\beta_1 \sqrt{M - S}}{2 \sqrt{u_1}} \frac{\partial V}{\partial S} = 0 \Rightarrow u_1^*(S) = (M - S)(0.5 \beta_i \frac{\partial V}{\partial S})^2. \tag{A7}\]
By substituting (A7) in (A6), we obtain an ODE in \(V(S)\), whose solution is
\[V(S) = \lambda_1 S + \lambda_{01}, \quad \text{where}\]
\[\lambda_1 = -2(\rho + \delta_i + \chi) + 2\sqrt{(\rho + \delta_i + \chi)^2 + (m_1 + \chi(1 - \phi)\beta_i^2}) \over \beta_i^2, \tag{A8}\]
\[\text{and}\]
\[\lambda_{01} = {M(\beta_1 \lambda_1)^2 \over 4(\rho + \chi)} + {\chi \lambda_{02} \over (\rho + \chi)}. \tag{A9}\]
The pre-crisis feedback strategy is \(u_1^*(S(t)) = (M(t) - S(t))(0.5 \beta_i \lambda_1)^2\), thus completing the proof. 

**Proof of Proposition 3 – Crisis Likelihood Effect**

We differentiate \(u_1^*(S)\) with respect to \(\chi\). Sign \(\frac{\partial u_1^*}{\partial \chi}\) is negative if
\[\lambda_2 < k = 2(-\rho - \delta_i + \sqrt{(\rho + \delta_i)^2 + m_i \beta_i^2}) / ((1 - \phi)\beta_i^2) \text{ and positive otherwise. Furthermore}
\[\frac{2(-\rho - \delta_i + \sqrt{(\rho + \delta_i)^2 + m_i \beta_i^2})}{\beta_i^2} > \frac{2(-\rho - \delta_i + \sqrt{(\rho + \delta_i)^2 + m_i \beta_i^2})}{\beta_i^2}, \text{which equals the long-term profit in the absence of crisis. Since a crisis diminishes the brand’s long-term profit}
\[\frac{2(-\rho - \delta_i + \sqrt{(\rho + \delta_i)^2 + m_i \beta_i^2})}{\beta_i^2} > \frac{2(-\rho - \delta_i + \sqrt{(\rho + \delta_i)^2 + m_i \beta_i^2})}{\beta_i^2} = \lambda_2. \text{ Hence } \lambda_2 < k,
\[\text{which implies that } \frac{\partial u_1^*}{\partial \chi} < 0.
\]

As \(\chi\) increases, \(u_1^*\) becomes smaller for every \(t\), so \(S(T^-)\) decreases, which, in turn, suppresses \(S(T^+)\) due to Equation (4). Consequently, the feedback advertising \(u_2^*\) increases (due to Proposition 2). Hence \(du_2^* / d\chi > 0\). 

**Proof of Proposition 4 – Damage Effect**
We differentiate $\frac{\partial u^*_1}{\partial \phi} = -\chi \lambda_2 / \sqrt{(\rho + \delta_1)^2 + (m_1 + \chi(1 - \phi))\beta_1^2}$, which shows that $\text{sign}(\frac{\partial u^*_1}{\partial \phi}) < 0$.

As $\phi$ increases, $u^*_1$ becomes smaller for every $t$, which reduces $S(T^-)$. In addition, $\phi$ directly lowers $S(T^+)$ due to Equation (4). These dual adverse effects suppress the sales trajectory $S(t)$ for $t > T$. Consequently, the feedback advertising via Proposition 2 increases $u^*_2$ as $\phi$ increases. Hence $du^*_2 / d\phi > 0$.

**Proof of Proposition 5 - Recovery**

The HJB equation for the crisis regime is,

$$(\rho + \omega)W(S) = \max_{u_2} \{ m_2 S - u_2 + \omega V((1 + \theta)S) + \frac{\partial W}{\partial S}(\beta_2 \sqrt{M - S - \delta_2 S})\},$$

and for the pre-crisis regime is

$$(\rho + \chi)V(S) = \max_{u_1} \{ m_1 S - u_1 + \chi W((1 - \phi)S) + \frac{\partial V}{\partial S}(\beta_1 \sqrt{M - S - \delta_1 S})\}.$$  \hspace{1cm} (A11)

Differentiating (A11) and (A12), we get the following FOCs:

$$u^*_1(S) = (M - S)(0.5 \beta_1 \frac{\partial V}{\partial S})$$

and

$$u^*_2(S) = (M - S)(0.5 \beta_2 \frac{\partial W}{\partial S}).$$

Substituting the FOCs back into (A11) and (A12), we obtain ODEs in $V(S)$ and $W(S)$, whose solutions are given by $V(S) = \lambda_1 S + \lambda_{01}$ and $W(S) = \lambda_2 S + \lambda_{02}$, where $(\lambda_1, \lambda_2, \lambda_{01}, \lambda_{02})$ are obtained from the algebraic equations:

$$\lambda_1 = \frac{-2(\rho + \delta_1 + \chi) + 2\sqrt{(\rho + \delta_1 + \chi)^2 + (m_1 + \chi(1 - \phi)\lambda_2)\beta_1^2}}{\beta_1^2},$$

$$\lambda_2 = \frac{-2(\rho + \delta_2 + \omega) + 2\sqrt{(\rho + \delta_2 + \omega)^2 + (m_2 + \omega(1 + \theta)\lambda_1)\beta_2^2}}{\beta_2^2},$$

$$\lambda_{01} = \frac{M(\beta_1 \lambda_1)^2}{4(\rho + \chi)} + \frac{\chi \lambda_{02}}{(\rho + \chi)},$$

and

$$\lambda_{02} = \frac{M(\beta_2 \lambda_2)^2}{4(\rho + \theta)} + \frac{\theta \lambda_{01}}{(\rho + \theta)}.$$  \hspace{1cm} (A15)

(A16)

Thus, suppressing $t$, the feedback strategies are $u_1^*(S) = (M - S)(0.5 \beta_1 \lambda_1)^2$ and $u_2^*(S) = (M - S)(0.5 \beta_2 \lambda_2)^2$.

**Proof of Proposition 6 - Duopoly**
In the second regime, the HJB equation for brand \( i = \{A, B\} \) is
\[
\rho W_i(S_A, S_B) = \text{Max}_{u_{i2}} \{m_{i2}S_i - u_{i2} + \sum_{i} \frac{\partial W_i}{\partial S_i} (\beta_{i2} \sqrt{u_{i2}} \sqrt{M - S_A - S_B - \delta_{i1} S_i})\}. \tag{A17}
\]

By differentiating (A17) wrt \( u_{i2} \), we obtain the FOC:
\[
-1 + \frac{\beta_{i2} \sqrt{M - S_A - S_B}}{2 \sqrt{u_{i2}}} \frac{\partial W_i}{\partial S_i} = 0
\]
\[
\Rightarrow u_{i2}^*(S_A, S_B) = (M - S_A - S_B)(0.5 \beta_{i2} \partial W_i / \partial S_i)^2.
\]

We substitute (A18) in (A17) to obtain a system of ODEs in \( W_i(S_A, S_B) \), whose solutions are
\[
\begin{bmatrix}
W_A(S_A, S_B) \\
W_B(S_A, S_B)
\end{bmatrix} =
\begin{bmatrix}
\lambda_{A2} & \lambda_{B2} \\
\mu_{A2} & \mu_{B2}
\end{bmatrix}
\begin{bmatrix}
S_A \\
S_B
\end{bmatrix} +
\begin{bmatrix}
\lambda_{02} \\
\mu_{02}
\end{bmatrix},
\tag{A19}
\]

where the coefficients are obtained from the algebraic equations:
\[
\rho \lambda_{A2} = \frac{1}{4} (4m_{A2} - 4\delta_{A2} \lambda_{A2} - \beta_{A2}^2 \lambda_{A2}^2 - 2\beta_{B2} \mu_{B2} \lambda_{B2}),
\tag{A20}
\]
\[
\rho \lambda_{B2} = \frac{1}{4} (-\beta_{A2}^2 \lambda_{A2}^2 - 4\delta_{B2} \lambda_{B2}^2 - 2\beta_{B2} \mu_{B2} \lambda_{B2}),
\tag{A21}
\]
\[
\rho \mu_{A2} = \frac{1}{4} (-\beta_{B2}^2 \mu_{B2}^2 - 4\delta_{A2} \mu_{A2}^2 - 2\beta_{A2} \lambda_{A2} \lambda_{A2}), \quad \text{and}
\tag{A22}
\]
\[
\rho \mu_{B2} = \frac{1}{4} (4m_{B2} - 4\delta_{B2} \mu_{B2} - \beta_{B2}^2 \mu_{B2}^2 - 2\beta_{A2} \lambda_{A2} \mu_{A2}).
\tag{A23}
\]

Thus, the post-crisis optimal feedback strategies are
\[
u_{A2}^*(S_A(t), S_B(t)) = (M(t) - S_A(t) - S_B(t))(0.5 \beta_{A2} \lambda_{A2})^2 \quad \text{and}
\]
\[
u_{B2}^*(S_A(t), S_B(t)) = (M(t) - S_A(t) - S_B(t))(0.5 \beta_{B2} \mu_{B2}).
\]

In the pre-crisis regime, the HJB equations are
\[
(\rho + \chi)V_i(S_A, S_B) = \text{Max}_{u_{i1}} \{m_{i1}S_i - u_{i1} + \chi W_i((1 - \phi_A)S_A, (1 - \phi_B)S_B) + \sum_{i} \frac{\partial V_i}{\partial S_i} (\beta_{i1} \sqrt{u_{i1}} \sqrt{M - S_A - S_B - \delta_{i1} S_i})\}. \tag{A24}
\]

By differentiating (A24) wrt \( u_{i1} \), we obtain the FOCs:
\[
-1 + \frac{\beta_{i1} \sqrt{M - S_A - S_B}}{2 \sqrt{u_{i1}}} \frac{\partial V_i}{\partial S_i} = 0
\]
\[
\Rightarrow u_{i1}^*(S_A, S_B) = (M - S_A - S_B)(0.5 \beta_{i1} \partial V_i / \partial S_i)^2.
\]

We substitute (A25) in (A24) to obtain a system of ODEs in \( V_i(S_A, S_B) \), whose solutions are
\[
\begin{bmatrix}
V_t(S_A, S_B) \\
V_t(S_A, S_B)
\end{bmatrix} =
\begin{bmatrix}
\lambda_{a1} & \lambda_{b1} \\
\mu_{a1} & \mu_{b1}
\end{bmatrix}
\begin{bmatrix}
S_A \\
S_B
\end{bmatrix} +
\begin{bmatrix}
\lambda_{a1} \\
\lambda_{b1}
\end{bmatrix},
\]

(A26)

where the coefficients are obtained from the algebraic equations:

\[
(\rho + \chi)\lambda_{a1} = \frac{1}{4}(4m_{a1} - 4\delta_{a1}\lambda_{a1} - \beta_{a1}^2\lambda_{a1}^2 - 2\beta_{b1}\mu_{b1}\lambda_{a1} + 4\chi(1 - \phi_a)\lambda_{a2}),
\]

(A27)

\[
(\rho + \chi)\lambda_{b1} = \frac{1}{4}(-\beta_{a1}\lambda_{b1}^2 - 4\delta_{b1}\lambda_{b1} - 2\beta_{b1}\mu_{b1}\lambda_{b1} + 4\chi(1 - \phi_b)\lambda_{b2}),
\]

(A28)

\[
(\rho + \chi)\mu_{a1} = \frac{1}{4}(-\beta_{b1}\mu_{b1}^2 - 4\delta_{a1}\mu_{a1} - 2\beta_{a1}\mu_{a1}\lambda_{a1} + 4\chi(1 - \phi_a)\mu_{a2}),
\]

(A29)

\[
(\rho + \chi)\mu_{b1} = \frac{1}{4}(4m_{b1} - 4\delta_{b1}\mu_{b1} - \beta_{b1}\mu_{b1}^2 - 2\beta_{a1}\lambda_{a1}\mu_{a1} + 4\chi(1 - \phi_b)\mu_{b2}).
\]

(A30)

Thus, the pre-crisis optimal advertising strategies are

\[
u^*_a(S_A(t), S_B(t)) = (M(t) - S_A(t) - S_B(t))(0.5\beta_{a1}\lambda_{a1})^2 \quad \text{and} \quad \nu^*_b(S_A(t), S_B(t)) = (M(t) - S_A(t) - S_B(t))(0.5\beta_{b1}\mu_{b1})^2.
\]

\textbf{Proof for Section 6.1 – Insulation Effect}

The HJB equation for the brand reads

\[
(\rho + \chi)V(S) = \max_{u_i} \{m_i S - u_i + \chi W((1 - (\phi_i + \frac{\phi_i}{S}))S) + \frac{\partial V}{\partial S} \beta_i \sqrt{u_i (\sqrt{M - S - \delta_i S})}\}.
\]

(A31)

Next, we solve for the pre-crisis optimal feedback strategies with size-dependent damage rate. Differentiating (A31) wrt \(u_i\), inserting the FOC into (A31), and solving the resulting ODE, we obtain the optimal feedback strategy,

\[
u^*_i(S) = (M - S)(0.5\beta_i \frac{\partial W_i}{\partial S})^2,
\]

(A32)

the value function \(W_i(S) = \lambda_i S + \lambda_{o1}\), where

\[
\lambda_i = \frac{-2(\rho + \delta_i + \chi) + 2\sqrt{(\rho + \delta_i + \chi)^2 + (m_i + \chi(1 - \phi_i)\lambda_{a1})\beta_i^2}}{\beta_i^2},
\]

(A34)

\[
\lambda_{o1} = \frac{M(\beta_i \lambda_{a1})^2}{4(\rho + \chi)} + \chi(\lambda_{a1} - \phi_i \lambda_{a2}) - \chi(\lambda_{o2} - \phi_i \lambda_{o2})}{(\rho + \chi)}.
\]

(A35)

\textbf{Proof for Section 6.2 – Investment in Quality}

If the manager chooses the low investment \(\gamma\) in quality, the HJB equation is
\[(\rho + \bar{\chi})V_L(S) = \max_{u_i} \{m_i S - u_i - v + \bar{\chi} W((1 - \phi)S) + \frac{\partial V_L}{\partial S} (\beta_i \sqrt{u_i \sqrt{M - S - \delta_i S}})\}. \tag{A36}\]

If s/he chooses the high investment \(\bar{V}\), the HJB equation is
\[(\rho + \bar{\chi})V_H(S) = \max_{u_i} \{m_i S - u_i - v + \bar{\chi} W((1 - \phi)S) + \frac{\partial V_H}{\partial S} (\beta_i \sqrt{u_i \sqrt{M - S - \delta_i S}})\}. \tag{A37}\]

Deriving the FOCs and solving the resulting ODEs, we find that under the high investment \(\bar{V}\), the value function is
\[V_H(S) = \lambda_{1H} S + \frac{\bar{\chi} \lambda_{\bar{V}02} - v}{\rho + \bar{\chi}} + \frac{M(\beta_i \lambda_{1H})^2}{4(\rho + \bar{\chi})}, \tag{A38}\]

where
\[\lambda_{1H} = \frac{-2(\rho + \delta_i + \bar{\chi}) + 2(\rho + \delta_i + \bar{\chi})^2 + (m_i + \bar{\chi}(1 - \phi_0)\lambda_2)\beta_i^2}{\beta_i^2}. \tag{A39}\]
\[\lambda_2 = \frac{-2(\rho + \delta_i) + 2(\rho + \delta_i)^2 + m_i \beta_2^2}{\beta_2^2}. \tag{A40}\]

For the low investment \(\bar{V}\), the value function is
\[V_L(S) = \lambda_{1L} S + \frac{\bar{\chi} \lambda_{\bar{V}02} - v}{\rho + \bar{\chi}} + \frac{M(\beta_i \lambda_{1L})^2}{4(\rho + \bar{\chi})}, \tag{A41}\]

where
\[\lambda_{1L} = \frac{-2(\rho + \delta_i + \bar{\chi}) + 2(\rho + \delta_i + \bar{\chi})^2 + (m_i + \bar{\chi}(1 - \phi_0)\lambda_2)\beta_i^2}{\beta_i^2}. \tag{A42}\]

It follows from Proposition 3 that \(\partial u_i / \partial \chi < 0\) because \(\partial \lambda_i / \partial \chi < 0\). That is, \(\lambda\) is a decreasing function of \(\chi\). Hence, \(\lambda_{1H} > \lambda_{1L}\) since \(\bar{\chi} < \bar{\chi}\). Given the different slopes of the value function, the manager chooses \(\bar{V}\) if \(V_H(S) > V_L(S)\), and \(\bar{V}\) otherwise. The crossover of the two value functions yields the sales threshold:
\[S^* = 0.5 \beta_i \left\{ \frac{\bar{\chi} \lambda_{\bar{V}02} - v}{\rho + \bar{\chi}} - \frac{\bar{\chi} \lambda_{\bar{V}02} - v}{\rho + \bar{\chi}} + \frac{M(\beta_i \lambda_{1H})^2}{4(\rho + \bar{\chi})} - \frac{M(\beta_i \lambda_{1L})^2}{4(\rho + \bar{\chi})} \right\} \]
\[\bar{\chi} - \bar{\chi} + (\rho + \delta_i + \bar{\chi})^2 + (m_i + \bar{\chi}(1 - \phi_0)\lambda_2)\beta_i^2 - \sqrt{\bar{\chi} + (\rho + \delta_i + \bar{\chi})^2 + (m_i + \bar{\chi}(1 - \phi_0)\lambda_2)\beta_i^2} + (m_i + \bar{\chi}(1 - \phi_0)\lambda_2)\beta_i^2. \]

Proof for Section 6.3 – Price and Advertising Competition

In the post-crisis regime, the HJB equations for the brands \(i = \{A, B\}\) are
\( \rho W_i(S_A, S_B) = \max \{(p_i - c_i)(1 - a_i p_i + b_i p_{-i})S_i - u_i \) \\
+ \sum_i \frac{\partial W_i}{\partial S_i} (\beta_i \sqrt{u_i \sqrt{M - S_A - S_B} - \delta_i S_i}) \).

By differentiating (A43) wrt to both the controls \( u_i \) and \( p_i \), we obtain the FOCs:

\[-1 + \frac{\beta_i \sqrt{M - S_A - S_B}}{2 \sqrt{u_i}} \frac{\partial W_i}{\partial S_i} = 0 \]
\[\Rightarrow \quad u_i^* (S_A, S_B) = (M - S_A - S_B)(0.5 \beta_i \frac{\partial W_i}{\partial S_i})^2, \]

and \( p_i = (1 + a_i c_i + b_i p_{-i}) / 2a_i \), which leads to the optimal prices:

\[p_{i2}^* = \frac{b_{i2} + a_{i2}(2 + 2a_{i2}c_i + b_{i2}c_i)}{4a_{i2}^2} - b_{i2}^2 \]

We substitute (A44) and (A45) in (A43) to obtain the system of ODEs in \( W(S_A, S_B) \), whose solutions are given by

\[\begin{bmatrix}
W_A(S_A, S_B) \\
W_B(S_A, S_B)
\end{bmatrix} = \begin{bmatrix}
\lambda_{A2} & \lambda_{B2} \\
\lambda_{A2} & \lambda_{A2}
\end{bmatrix} \begin{bmatrix}
S_A \\
S_B
\end{bmatrix} + \begin{bmatrix}
\lambda_{02} \\
\lambda_{02}
\end{bmatrix}, \]

where the coefficients are obtained from the algebraic equations:

\[\rho \lambda_{A2} = \frac{a_{i2}^2(1 + b_{i2}c_i + a_{i2}(2 - 2a_{i2}c_i + b_{i2}c_i))^2}{4a_{i2}b_{i2}^2} - \frac{1}{4} (4\delta_{A2}^2 \lambda_{A2} + \beta_{A2}^2 \lambda_{A2}^2 + 2\beta_{B2}^2 \lambda_{A2} \lambda_{B2}), \]

\[\rho \lambda_{B2} = \frac{1}{4} (\beta_{A2}^2 \lambda_{A2}^2 - 4\delta_{B2}^2 \lambda_{B2}^2 - 2\beta_{B2}^2 \mu_{A2} \lambda_{B2}), \]

\[\rho \mu_{A2} = \frac{1}{4} (-\beta_{B2}^2 \mu_{A2}^2 - 4\delta_{A2}^2 \mu_{A2}^2 - 2\beta_{A2}^2 \lambda_{A2} \mu_{A2}), \]

\[\rho \mu_{B2} = \frac{a_{i2}^2(1 + b_{i2}c_i + a_{i2}(2 - 2a_{i2}c_i + b_{i2}c_i))^2}{4a_{i2}b_{i2}^2} - \frac{1}{4} (4\delta_{B2}^2 \mu_{B2}^2 + \beta_{B2}^2 \mu_{B2}^2 + 2\beta_{A2}^2 \lambda_{A2} \mu_{A2}). \]

Thus, the post-crisis feedback advertising strategies are

\[u_{i2}^* (S_A(t), S_B(t)) = (M(t) - S_A(t) - S_B(t))(0.5 \beta_{i2} \lambda_{i2})^2 \]

\[u_{i2}^* (S_A(t), S_B(t)) = (M(t) - S_A(t) - S_B(t))(0.5 \beta_{i2} \mu_{i2})^2 \]

In the pre-crisis regime, the HJB equations are

\[(\rho + \chi)W_i(S_A, S_B) = \max \{(p_i - c_i)(1 - a_i p_i + b_i p_{-i})S_i - u_i \) \\
+ \sum_i \frac{\partial W_i}{\partial S_i} (\beta_i \sqrt{u_i \sqrt{M - S_A - S_B} - \delta_i S_i}) \].
By differentiating (A50) wrt \( u_{i1} \), we obtain the FOCs:

\[
-1 + \frac{\beta_i \sqrt{M - S_A - S_B}}{2u_{i1}} \partial V_i / \partial S_i = 0
\]

\[
\Rightarrow u_{i1}^*(S_A, S_B) = (M - S_A - S_B)(0.5 \beta_i \partial V_i / \partial S_i)^2,
\]

and \( p_{i1} = (1 + a_{i1}c_i + b_{i1}p_{3-i,1}) / 2a_{i1} \), which yields the optimal prices:

\[
p_{i1}^* = \frac{b_{i1} + a_{i1}(2 + 2a_{i1}c_A + b_{i1}c_B)}{4a_{i1}a_{b1} - b_{i1}b_{b1}} \quad \text{and} \quad p_{b1}^* = \frac{b_{b1} + a_{b1}(2 + 2a_{b1}c_B + b_{b1}c_A)}{4a_{a1}a_{a1} - b_{a1}b_{a1}}.
\]

We substitute (A51) and (A52) in (A50) to obtain the system of ODEs in \( V(A, S_B) \), whose solutions are given by

\[
\begin{bmatrix}
V_A(S_A, S_B) \\
V_B(S_A, S_B)
\end{bmatrix} = \begin{bmatrix}
\lambda_{a1} & \lambda_{b1} \\
\mu_{a1} & \mu_{b1}
\end{bmatrix} \begin{bmatrix}
S_A \\
S_B
\end{bmatrix} + \begin{bmatrix}
\lambda_{a1} \\
\mu_{a1}
\end{bmatrix},
\]

where the coefficients are obtained from the algebraic equations:

\[
(\rho + \chi)\lambda_{a1} = \frac{a_{a1}(b_{a1}(1 + b_{a1}c_A) + a_{b1}(2 - 2a_{a1}c_A + b_{a1}c_B))^2}{(b_{a1}b_{b1} - 4a_{a1}a_{b1})^2}
\]

\[
-4\delta_{a1}\lambda_{a1} + \beta_{a1}^2\lambda_{a1}^2 + 2\beta_{b1}^2\mu_{b1}\lambda_{b1} - 4\chi(1 - \phi_A)\lambda_{a1},
\]

\[
(\rho + \chi)\lambda_{b1} = \frac{1}{4}(-\beta_{b1}^2\lambda_{b1}^2 - 4\delta_{b1}\lambda_{b1} - 2\beta_{b1}^2\mu_{b1}\lambda_{b1} + 4\chi(1 - \phi_B)\lambda_{b1}),
\]

\[
(\rho + \chi)\mu_{a1} = \frac{1}{4}(-\beta_{a1}^2\mu_{a1}^2 - 4\delta_{a1}\mu_{a1} - 2\beta_{a1}^2\mu_{a1}\lambda_{a1} + 4\chi(1 - \phi_A)\mu_{a1}),
\]

\[
(\rho + \chi)\mu_{b1} = \frac{a_{b1}(b_{b1}(1 + b_{b1}c_B) + a_{b1}(2 - 2a_{b1}c_B + b_{b1}c_A))^2}{(b_{a1}b_{b1} - 4a_{a1}a_{b1})^2}
\]

\[
-4\delta_{b1}\mu_{b1} + \beta_{b1}^2\mu_{b1}^2 + 2\beta_{b1}^2\mu_{b1}\lambda_{a1} - 4\chi(1 - \phi_B)\mu_{b1}.
\]

Thus, the pre-crisis feedback advertising strategies are

\[
\begin{align*}
u_{a1}^*(S_A(t), S_B(t)) &= (M(t) - S_A(t) - S_B(t))(0.5 \beta_{a1}\lambda_{a1})^2 \quad \text{and} \\
u_{b1}^*(S_A(t), S_B(t)) &= (M(t) - S_A(t) - S_B(t))(0.5 \beta_{b1}\mu_{b1})^2.
\end{align*}
\]


