SPATIOTEMPORAL ALLOCATION OF ADVERTISING BUDGETS

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How should brand managers determine the optimal advertising budget to generate sales and maximize profit from multiple regions and over time? How much of it should be set aside for national advertising? How should they allocate the rest across multiple regions? This paper addresses these questions by developing a method for optimal allocation of resources based on an empirically validated model of how national and regional advertising generate sales over time. The authors derive the profit-maximizing total budget, its optimal split between national and regional spends, and its optimal allocation across multiple regions. To this end, they formulate a spatiotemporal model that accounts for spatial and serial dependence, spatial heterogeneity, neighborhood effects and sales dynamics. Because of spatial and serial dependence, correlated multivariate Brownian motion drives the sales dynamics, resulting in a second-order differential equation for the Hamilton-Jacobi-Bellman (HJB) equation with multiple states (i.e., regional sales) and multiple controls (i.e., regional and national advertising expenditures). By solving the HJB equation analytically, the authors furnish closed-form expressions for the optimal total budget and its regional allocations. In addition, they develop a method to estimate the proposed model and apply it to market data from a leading German cosmetics company. Using the estimated parameters, they evaluate the optimal budget and allocations. Comparing them with actual company policy, the proposed approach enhances profit by 5.07%. Finally, the proposed method not only identifies which regions under- or over-spend, but also reveals how much budget to shift from national to regional advertising (or vice versa).

**Keywords:** Advertising, Budget Allocation, Spatiotemporal Model, Optimal Allocation, Neighborhood Effects, Spatial Dependence, Spatial Heterogeneity
In our recent meeting with the chief marketing officer of a leading cosmetics firm, she broached the topic of how to spend a hundred million Euros to advertise a brand, whether 100 million is the “right” sum, how much of it should be set aside for national advertising, and how to allocate it across the seven Nielsen regions of Germany? When we asked what they do now, she revealed (see Figure 1) the actual allocation as well as the spending plan based on marketing textbooks, which relies on brand development index (BDI) as the basis for allocation, though, she noted that this BDI plan recommends neither the total sum, nor how much to spend on national ads, let alone whether it is optimal.

[Insert Figure 1]

The BDI-based approach results in advertising spend proportional to the per capita sales in each region. To assess the optimality of these allocation decisions, we require both the response model and profit function, which the BDI-based approach lacks, a point alluded by Lodish (2007, p. 24). This drawback highlights the need for a method for optimal allocation of resources based on (i) an empirically validated model of how national and regional advertising generates sales over time and (ii) a normative analysis that derives the profit-maximizing total budget, its optimal split between national and regional spends, and its optimal allocation across the multiple regions. This paper develops a method to answer the above questions.

Previous research builds spatial models to capture variations in brand performance across regions (e.g., Ataman, Mela, and van Heerde 2007; Bell, Ho, and Tang 1998; Bronnenberg and Mahajan 2001; Bronnenberg, Dhar, and Dubé 2007a and 2007b; Chan, Padmanabhan, and Seetharaman 2007; Thomadsen 2007). These models account for spatial heterogeneity (i.e., marketing response differs across regions), neighborhood effects (i.e., past sales in neighboring regions affect the focal region) and spatial dependency (i.e., errors are correlated across regions),
but ignore the dynamic effects of advertising. To account for dynamics, spatiotemporal models have been recently introduced in marketing (e.g., Albuquerque, Bronnenberg, and Corbett 2007; Bell and Song 2007; Choi, Hui, and Bell 2010), which, however, do not provide closed-form budgeting or allocation expressions. In contrast, several studies (e.g., Doyle and Saunders 1990; Naik and Raman 2003; Skiera and Albers 1998) that provide normative findings disregard the spatial effects, which when ignored lead to inaccurate forecasts (Giacomini and Granger 2004). Thus, as the literature review shows, no study estimates a spatiotemporal model of advertising and derives the optimal budget and allocation, accounting for spatial and serial dependence, spatial heterogeneity, neighborhood effects and sales dynamics.

To fill this gap, we formulate a spatiotemporal model which explicitly distinguishes between national and regional advertising effects. While national advertising offers an efficient way to build sales globally, regional advertising allows managers to enhance sales locally. We capture observed dependencies across neighboring regions via neighborhood effects; and we capture unobserved dependencies across neighboring regions and across contiguous time periods via spatial correlation and serial correlation, respectively. Because of the unobserved spatial and serial dependencies, we obtain correlated multivariate Brownian motion affecting the sales dynamics. Consequently, the resulting Hamilton-Jacobi-Bellman equation is a second-order differential equation with multiple states (i.e., regional sales) and multiple controls (i.e., regional and national advertising expenditures). Nonetheless, we solve it analytically and thus derive the closed-form expressions for the optimal total budget and its regional allocations (see Propositions 1 and 2). This normative contribution not only furnishes the optimal budget and allocation simultaneously as recommended in the literature (e.g., see Mantrala, Sinha and Zoltners 1992), but also is novel to spatiotemporal literature in marketing and economics.
Beside normative analysis, we extend the estimation method of Baltagi, Song, Jung, and Koh (2007) by incorporating sales dynamics, neighborhood effects and spatial heterogeneity. We then empirically validate the proposed model by using market data from a leading German cosmetics company. The results indicate good fits for both in-sample and out-of-sample data. The coefficients for spatial and serial dependence are statistically significant. Using the estimated parameters, we evaluate the optimal budget and allocation. Comparing them with actual spends and BDI-based recommendations, we find misallocations at both the national and regional levels. The total budget should be reduced from €7.9 million to €5.9 million per month, and its split to national versus regional advertising should be changed from 92.4% to 85.9%. Furthermore, compared to actual profit, BDI-based allocations increase profit by about 0.37%, whereas the proposed approach increases profit by 5.07%. More importantly, it provides a systematic way to assess whether a specific region under-spends (e.g., region 2 should increase spending by 63%) or over-spends (e.g., region 7 should decrease spending by 83%). Such optimality assessments add diagnostic value for the chief marketing officer. Why? Because she learns whether to shift the budget from national to regional advertising (or vice versa) and how to allocate resources across the different regions.

We organize the paper as follows. We first review the literature streams on BDI-based allocations, spatiotemporal models, and optimal allocation studies. Then, we propose a spatiotemporal model, derive normative results, and develop an estimation method. Next, we illustrate an empirical application and furnish substantive findings. We close the paper by discussing the optimal allocations under time-varying parameters, the value of continuous-time analysis, the effects of discount rate, and pulsing versus even spending.
Brand Development Index

Brand development index (BDI) indicates how brand’s sales perform relative to the size of the consumer market. Specifically, to obtain BDI of a given region, we first determine (i) the brand sales in a given region as a ratio to the national sales and (ii) the region’s population as a ratio to the national population; then $BDI = 100 \times \frac{\text{fraction in (i)}}{\text{fraction in (ii)}}$. In words, a BDI score of 100 means that the region’s sales are on par with the sales expected for its size; scores above (below) 100 indicate over- (under-) performing regions. For example, BDI score of 120 means brand sales in that region is 20% more than what would be expected from that region’s market size; BDI score of 75 points means its brand sales are 25% below the expected sales given its market size. A similar definition for category sales yields the category development index (CDI), which tends to be correlated with the BDI scores.

Brand managers prioritize various regions using the BDI scores. Advertising textbooks (see, e.g., Goodrich and Sissors 1996, p. 43; Hiebing and Cooper 2004, p. 247; Sissors and Baron 2002, p. 182) recommend allocating budgets proportional to the BDI scores. For example, consider a firm that operates in two regions A and B with sales fraction of 0.7 and 0.3, population shares of 0.4 and 0.6, and the regional budget of one million dollars. Then $BDI_A = 100 \times 0.7/0.4 = 175$ and $BDI_B = 100 \times 0.3/0.6 = 50$. So the resulting allocation to region A = $(175/225) \times 1,000,000 = $777,777 and to region B = $(50/225) \times 1,000,000 = $222,223. This example highlights that the BDI approach does not yield the spending on national advertising (and hence the total budget). Moreover, we do not know whether these allocations are optimal. Why? Because the optimal allocation depends on the sales lift resulting from the incremental ad
spending (Abraham and Lodish 1990), whereas the BDI approach does not quantify this marginal effect of advertising. To this end, we need to formulate spatial models, which we review next.

Spatial and Spatiotemporal Models with Marketing Applications

Bradlow et al. (2005) and Bronnenberg (2005) provide extensive reviews of spatial models in marketing. In Table 1, we complement their works by comparing nineteen studies based on the following features: spatial heterogeneity, neighborhood effects, spatial dependence, serial dependence, sales dynamics, marketing decision variables, and optimal decisions.

Table 1 shows that most applications focus on pricing rather than advertising (e.g., Greenhut 1981; Jank and Kannan 2005; Pinske, Slade, and Brett 2002), while others examine spatial diffusion and demand patterns (e.g., Bell and Song 2007; Bronnenberg, Dhar, and Dubé 2007a; Duan and Mela 2009; Gatignon, Eliashberg, and Robertson 1989). Bhargava and Donthu (1999) is a notable exception for investigating the spatial effects of billboard effectiveness using field experiments. Moreover, most studies analyze spatial data for fast moving consumer goods in food categories in the US markets. In contrast, this study analyzes spatial advertising data from non-food product category (viz., cosmetics) in a major non-US market (viz., Germany).

Three types of spatial effects considered in the literature are spatial heterogeneity, neighborhood effects and spatial dependence. Spatial heterogeneity occurs due to the non-uniform effects of space, for example, due to differences in urban growth, unequal populations, differential incomes, or differences in media consumption—all of which might reflect in different advertising effectiveness across regions (see Anselin 1988, p. 11-15; Bradlow et al. 2005). To account for spatial heterogeneity, parameters differ across regions (i.e., we estimate
region-specific parameters). Neighborhood effects arise because sales in a region depend not only on past sales in that region but also on past sales in neighboring regions. These effects can materialize due to various factors, for example—due to communication between individuals in different regions, cross-regional travel and business links, similar marketing programs and retail landscapes, or passive observation of products in other regions (Bell and Song 2007; Yang and Allenby 2003). On the other hand, spatial dependency represents the co-variation of observations across spatial units. It arises due to the effects of unobserved similarities between regions based on their socio-economic makeup, usage of resources, or physical characteristics (Ter Hofstede, Wedel, and Steenkamp 2002). To account for spatial dependency, error terms are correlated across regions via a contiguity matrix.

Table 1 reveals that most studies incorporate neighborhood effects and spatial dependence. However, several studies do not allow response parameters to vary across regions (see the column for spatial heterogeneity). Like spatial dependence, auto-correlation relaxes the assumption of independence among observations over time. Such serial dependence has been studied by Bronnenberg and Mela (2004), Bronnenberg and Sismeiro (2002), and Bronnenberg and Mahajan (2001). As for sales dynamics, diffusion models incorporate it by design (e.g., Bell and Song 2007). Bronnenberg and Sismeiro (2002) account for dynamic shocks in price effects, while Jank and Kannan’s (2006) online learning choice model makes spatial predictions based on spatially dispersed consumer choices observed up to the previous period. However, no study considers the five factors—spatial heterogeneity, neighborhood effects, spatial dependence, serial dependence, and sales dynamics—in the context of advertising.

Finally, we survey the type and scope of normative analyses. In the diffusion context, Choi, Hui, and Bell (2010) use numerical simulation to assess the impact of different imitation
strategies, but do not derive optimal decisions analytically. Similarly, Duan and Mela (2009) explore the role of spatial demand on outlet location and suggest desired locations for additional outlets numerically. Chan, Padmanabhan, and Seetharaman (2007) investigate numerically the impact of a potential merger between gasoline retail firms on the local markets. Thomadsen (2007) studies product positioning for asymmetric firms in spatial markets, but computes equilibrium strategies numerically. Taken together, no study analytically derives optimal marketing strategies for dynamic and spatially related markets. Thus, this study augments the marketing literature.

**Optimal Allocation of Advertising Budgets**

We review a few prominent studies that offer normative advertising analyses; as we shall see, they ignore either neighborhood effects or spatial dependence or sales dynamics. The classical study by Nerlove and Arrow (1962) envisions advertising to build a stock of goodwill that depreciates over time, and they obtain the dynamically optimal advertising strategy. We could generalize this solution for every region individually and then sum it up to determine the total advertising budget. However, the resulting budget would ignore the spatial effects (i.e., spatial heterogeneity, the neighborhood effects, and spatial dependence). Doyle and Saunders (1990) develop an approach for multi-product advertising budgeting that accounts for advertising spillovers across multiple products. However, we cannot transfer their multi-product solution to our multi-region setting because products, unlike regions, do not exhibit spatial proximity (e.g., effects of neighboring regions). Skiera and Albers (1998) maximize profits obtained from multiple sales territories by optimizing travel costs. Although they incorporate spatial heterogeneity in sales potential, they ignore sales dynamics, neighborhood effects and spatial dependence. More recently, Naik and Raman (2003) investigate the allocation for multi-media
advertising in a dynamic setting. They prove a counter-intuitive result that, as cross-media synergy increases, managers should increase the total budget and that this incremental budget should be allocated inversely proportional to the media effectiveness, i.e., the less (more) effective medium gets the larger (smaller) share of the incremental budget. Akin to Doyle and Saunders (1990), the multiple media do not exhibit spatial proximity, and so they also ignore the spatial and neighborhood effects. Hence, there exists a gap in the extant literature on spatiotemporal allocation of national and regional advertising budgets. To fill this gap, we next formulate a dynamic advertising model with spatial effects and then address budget allocation and model estimation.

**MODEL FORMULATION**

*Sales Dynamics, Neighborhood Effects, and Spatial Heterogeneity*

Marketing research has shown that consumers’ response varies across both time (e.g. Naik and Raman 2003) and regional markets (e.g., Bhargava and Donthu 1999; Lodish 2007) and due to inter-regional effects (Bell and Song 2007). Hence, we incorporate sales dynamics via carryover effects, neighborhood effects through the impact of lagged sales in neighboring regions, and spatial heterogeneity by allowing region-specific parameters. For each region $i$, the local region’s sales depend on its own and neighboring regions’ lagged sales as well as both the regional and national advertising, which we express as follows:

$$S_{it} = \lambda_i S_{it-1} + \beta_i \sqrt{R_{it}} + \alpha_i \sqrt{N_t} + \sum_{j=1}^{K} \gamma_{ij} S_{jt-1} + \epsilon_{it}, \quad i, j = 1, \ldots, K, \quad t = 1, \ldots, T,$$

(1)

where $S_{it}$ measures the units sold in region $i$ at time $t$, $S_{jt-1}$ denotes the sales in neighboring region $j$ at time $t-1$, $R_{it}$ and $N_t$ are the regional and national dollars spent on advertising, and the error terms $\epsilon_i = (\epsilon_{it}, \ldots, \epsilon_{Kt}) \sim N(0, \Sigma_{\epsilon})$. The region-specific parameters $\lambda_i$ are the regional
carryover effects; $\gamma_{ij}$ quantifies the spatial effects of neighboring region $j$’s lagged sales; $\beta_i$ measures regional ad effectiveness; and $\alpha_i$ measures the effect of national advertising on local sales. Because the marketing mix differs between regional (direct mail, local newspapers, radio) and national (TV, magazines, national newspapers) advertising, the respective advertising response effects ($\beta_i$ and $\alpha_i$) also vary. To capture spatial effects, we specify inter-regional dependence between contiguous regions via the contiguity matrix $\tilde{C}$, whose elements $\tilde{c}_{ij}$ equals 1 if the region $i$ shares its border with the region $j$ and zero otherwise — see Panel A of Figure 2 for the seven Nielsen regions of Germany and its Panel B for the corresponding contiguity matrix $\tilde{C}$. We standardize this contiguity matrix such that each row sums to unity and denote the resulting matrix by $C$ (and its elements by $c_{ij}$). Hence the neighborhood effects $\gamma_{ij}$ equal $\gamma c_{ij}$, where $\gamma$ represents the overall neighborhood effect. Consistent with the prior literature (e.g., Chintagunta 1993), the square-roots in equation (1) incorporate the notion of diminishing returns, which means that the more we spend on advertising the less incremental sales we get. The next subsection introduces spatial and serial dependence in the error term $\varepsilon_{it}$.

Spatial and Serial Dependence

Several factors not explicitly included in the model and thus relegated to the error terms introduce spatial dependence (Bradlow et al. 2005; Bronnenberg 2005; Chintagunta, Dubé, and Goh 2005). Such unobserved factors emerge because neighboring “regions … often share climate, resources, history and sociodemographic and economic makeup.” (Ter Hofstede, Wedel, and Steenkamp 2002, p.161). As these factors do not explicitly enter the model, we allow the
error terms to be correlated across regions. Given the standardized contiguity matrix $C$, we express the error term $\varepsilon_i$ as follows:

$$
\varepsilon_i = \mu \sum_{j=1}^{K} c_{ij} \varepsilon_j + \eta_i,
$$

(2)

where $i$ and $j$ denote the various regions, $\mu$ is the spatial correlation, and the error vector $\eta_i = (\eta_i, \ldots, \eta_{K_i})'$ follows $N(0, \Sigma)$. Equation (2) allows for neighboring regions’ shock to affect the focal region. Because this shock does not persist over time, we extend the error structure to incorporate serial dependence.

Spatial dependence reflects disturbances between regions; serial dependence captures shocks within a region over time. That is, the unobserved shocks within a region may carry over to subsequent periods (e.g., Bronnenberg and Mahajan 2001; Bronnenberg and Mela 2004). Such shocks could occur due to unobserved consumer or manufacturer behavior or due to actions by other unobserved participants like distributors or retailers (Bronnenberg and Sismeiro 2002). To incorporate such serial dependence, we modify the error structure in (2) by allowing $\eta_i$ to be serially correlated with $\eta_{i,t-1}$ as follows:

$$
\eta_i = \omega \eta_{i,t-1} + \nu_i,
$$

(3)

where $\omega$ represents the serial dependence and $\nu_i = (\nu_i, \ldots, \nu_{K_i})' \sim N(0, \Sigma_{\nu})$. In the next section, we derive closed-form expressions for optimal advertising budget, its optimal split between national and regional spends, and its optimal allocation across multiple regions.

**SPATIOTEMPORAL ALLOCATION**

*Continuous-time Dynamics and Uncertainty*
To facilitate the derivation of optimal budget and allocations, we first convert the
discrete-time model to its continuous-time analog (see Malliaris and Brock 1982, p. 66-68).

Specifically, we re-write equation (1) as
\[ dS_i(t) = (\delta_i S_i(t) + \beta_i \sqrt{R_i(t)} + \alpha_i \sqrt{N(t)} + \sum_{j \neq i}^{K} \gamma_{ij} S_j(t))dt \]
\[ + \sum_{j=1}^{K} \tilde{\sigma}_{ij} d\tilde{W}_j(t), \] where \( \delta_i = (1 - \lambda_i) \), and \( d\tilde{W}_j \) is a random process whose properties we derive in
the Web Appendix to characterize the explicit dependence of \( \tilde{\sigma}_{ij} \) on the spatial and serial
dependence parameters (\( \mu \) and \( \omega \)). Thus, denoting \( f_i = -\delta_i S_i + \beta_i \sqrt{R_i} + \alpha_i \sqrt{N} + \sum_{j \neq i}^{K} \gamma_{ij} S_j \) for
\( i = 1, \ldots, K \), the system of regional continuous-time stochastic sales differential equations is,
\[ dS_i = f_i dt + \sum_{j=1}^{K} \tilde{\sigma}_{ij} d\tilde{W}_j. \] \hspace{1cm} (4)

**Long-term Future Profit Expectation**

Next, to evaluate the long-term future profit resulting from (4), we let \( \rho \) be the discount
rate and \( m_i \) represent the margins from the regions \( i \). Then the long-term expected future profit is
\[ J(R_1, \cdots, R_K, N) = E \left( \int_{0}^{\infty} e^{-\rho t} \left[ \sum_{i=1}^{K} m_i S_i - \sum_{i=1}^{K} R_i - N \right] dt \right), \] \hspace{1cm} (5)
where \( E(\cdot) \) denotes the expectation, and the integral sums up the discounted future operating
profit, which equals the gross profit less the total ad spends (shown in the square brackets).

In equation (5), the expected profit \( J(\cdot) \) depends on the spending levels \( R_i \) and \( N \). A
meager ad spending \((R_i, N)\) would generate limited sales and earn small profit; as \((R_i, N)\)
increases, the sales \( S_i \) increases and enhances the profit \( J \); however, beyond a certain level,
further advertising increases sales but with a diminishing rate and thus the profit \( J \) decreases.

Given this inverted-U shape of \( J(\cdot) \) with respect to its arguments, managers should operate at the
“sweet spot” where the regional and national advertising are neither too little nor too much. To this end, they should find the optimal regional advertising $R_i^*$ and the optimal national advertising $N^*$ by maximizing the long-term future profit in (5) subject to the stochastic dynamic evolution in (4) that incorporates sales dynamics, neighborhood effects, spatial heterogeneity, and spatial and serial dependence.

**Optimal Budget, Allocations, and Split**

To maximize $J(R_1, \cdots, R_K, N)$, we define the value function, which represents the largest attainable profit when we optimally set all the regional and national advertising spends. Let the value function $V(S_1, \cdots, S_K) = \text{Max}(J(R_1^*, \cdots, R_K^*, N^*))$. Then the value function satisfies the stochastic Hamilton-Jacobi-Bellman (HJB) equation:

$$\rho V = \text{Max}[(\sum_i m_i S_i - \sum_i R_i - N) + \sum_i V_i f_i + \frac{1}{2} \text{Tr}(\Sigma V)]$$

where $V_i = \frac{\partial V}{\partial S_i}$, the $K \times K$ matrix $\hat{V}$ consists of the elements $\{v_{ij}\}$, $v_{ij} = \frac{\partial^2 V}{\partial S_i \partial S_j}$, and $\text{Tr}(\cdot)$ denotes the trace of a matrix.

Equation (6) is a second-order differential equation for the value function $V$, and it constitutes the necessary and sufficient condition for an optimum (Lewis 1986, p. 298; Sethi and Thompson 2000, p. 345-7). Its first term on the right hand side captures the immediate profit from the current states (i.e., sales) and controls (i.e., national and regional advertising). The second term incorporates the change in future profit due to changes in sales trajectories induced by current advertising decisions. The last term reflects the profit consequences arising from the spatial and serial dependence.
By analytically solving the stochastic HJB equation (6) in the Web Appendix, we obtain the closed-form expressions for the optimal regional and national spends $R_i^*$ and $N^*$, which we next present.

**Proposition 1:** For each region $i$, the optimal regional advertising expenditure is

$$R_i^* = \frac{1}{4} \left( \beta_i b_i \right)^2,$$

the optimal national advertising expenditure equals

$$N^* = \frac{1}{4} \left( \alpha_i b_i + \cdots + \alpha_k b_k \right)^2,$$

where

$$\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_k \end{bmatrix} = \begin{bmatrix} \rho + \delta_1 & -\gamma_{11} & \cdots & -\gamma_{1k} \\ -\gamma_{21} & \rho + \delta_2 & \cdots & -\gamma_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ -\gamma_{k1} & -\gamma_{k2} & \cdots & \rho + \delta_k \end{bmatrix}^{-1} \begin{bmatrix} m_1 \\ m_2 \\ \vdots \\ m_k \end{bmatrix}.$$  

Hence the optimal total budget and its optimal split are given by

$$B^* = N^* + \sum_{i=1}^K R_i^*,$$

$$\phi^* = \frac{N^*}{B^*}.$$

**Proof:** See Web Appendix.

We note that the optimal allocations in (7) and (8) depend on the neighborhood effects $\gamma_{ij}$ through $b_i$. Remarkably, the optimal allocations do not depend on the spatial and serial correlations ($\mu$ and $\omega$). Why? Because managers do not control the uncertainty in realized sales due to spatial and serial effects. Nonetheless, the realized sales influence the estimated parameter values and their efficiency (see Equations 12 and 13). Hence, the effects of spatial and serial correlations manifest themselves in the estimated allocations and their operating range.
This discussion completes the optimal budgeting and allocation of dynamic spatiotemporal models. To apply the above formulae in practice, managers need to obtain parameter values by estimating the model equations (1)–(3) using market data. Hence, we develop an estimation method in the next section.

**MODEL ESTIMATION**

Recently, Baltagi et al. (2007) provide a method that accounts for spatial and serial dependence. Their method groups observations of all \( K \) regions together and then stacks these groups for \( t = 1, \cdots, T \) time periods (ibid, see p. 7). This way of stacking is not appropriate when estimating the proposed model for three reasons. First, to accommodate sales dynamics, we need to keep contiguous time periods grouped together. Second, to incorporate spatial heterogeneity, we need to create a block diagonal structure (rather than stacked) for the regressor matrix. Third, to include neighborhood effects, we stack the regressor matrix with a column formed by composite variables of lagged neighboring region sales. These features result in a new error covariance matrix than differs from Equation (2.12) in Baltagi et al. (2007). To obtain the proper error covariance matrix, we construct new vectors and matrices.

Let \( \theta_i = (\lambda_i, \beta_i, \alpha_i)' \) denote the parameters for each region \( i \), and \( Y_i = (S_{i2}, \cdots, S_{iT})' \) be the \((T - 1) \times 1\) vector of sales starting from the period 2 through \( T \) (because the lag operation creates a missing value) for each region \( i \). We then create a matrix \( X_i \) of dimension \((T - 1) \times 3\) by stacking \((S_{i,t-1}, R_u, N_u)'\) from \( t = 2, \cdots, T \). We generate a composite vector \( \tilde{X}_i = (\tilde{X}_{i2}, \cdots, \tilde{X}_{iT})' \) of length \((T - 1)\), where \( \tilde{X}_u = \sum_{j = i+1}^K c_j S_{j,t-1} \) for \( t = 2, \cdots, T \). Using these vectors and matrices, we convert equation (1) into the following system:
\[
\begin{bmatrix}
Y_1 \\
\vdots \\
Y_K
\end{bmatrix} = \begin{bmatrix}
X_1 & 0 & \cdots & \tilde{X}_1 \\
0 & \ddots & \cdots & 0 \\
\vdots & \cdots & \ddots & \vdots \\
0 & \cdots & 0 & X_K
\end{bmatrix} \begin{bmatrix}
\theta_1 \\
\vdots \\
\theta_K \\
\gamma
\end{bmatrix} + \begin{bmatrix}
\varepsilon_1 \\
\vdots \\
\varepsilon_K
\end{bmatrix},
\tag{10}
\]

where \( \varepsilon_i = (\varepsilon_{i1}, \ldots, \varepsilon_{iT})' \). Thus, we re-express the above equation as

\[
Y = X\theta + \varepsilon,
\tag{11}
\]

where \( Y = (Y_1', \ldots, Y_K')' \), \( \theta = (\theta_1', \ldots, \theta_K', \gamma)' \), \( \varepsilon = (\varepsilon_1', \ldots, \varepsilon_K')' \), and the matrix \( X \) constructed as shown in (10).

For parameter estimation, we use (12) and (13) below:

\[
\hat{\theta} = (X'\tilde{\Sigma}_e^{-1}X)^{-1}X'\tilde{\Sigma}_e^{-1}Y, \quad \text{and}
\tag{12}
\]

\[
\tilde{\Sigma}_e(\mu, \omega) = ((I - \mu C)^{-1}\Sigma_e(I - \mu C)^{-\omega} \otimes M(\omega)),
\tag{13}
\]

where \( M(\omega) = \frac{1}{(1-\omega^2)} \begin{bmatrix}
1 & \omega & \omega^2 & \cdots & \omega^{r-2} \\
\omega & 1 & \omega & \cdots & \omega^{r-3} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\omega^{r-2} & \omega^{r-3} & \omega^{r-4} & \cdots & 1
\end{bmatrix} \)

and \( \otimes \) indicates Kronecker product. We derive (13) in the Web Appendix. For statistical inference, we obtain the standard errors via the squared-root of the diagonal of the matrix \((X'\tilde{\Sigma}_e^{-1}X)^{-1}\). For robust inference, we obtain standard errors from the matrix

\[
(X'\tilde{\Sigma}_e^{-1}X)^{-1}[X'\tilde{\Sigma}_e^{-1}P\tilde{\Sigma}_e^{-1}X](X'\tilde{\Sigma}_e^{-1}X)^{-1},
\]

where \( P \) is a diagonal matrix of squared residuals (for details, see MacKinnon and White 1985). For spatiotemporal estimation, we maximize the concentrated log-likelihood function \( LL(\mu, \omega) = 0.5[Ln(|\tilde{\Sigma}_e|) + \hat{\varepsilon}'\tilde{\Sigma}_e^{-1}\hat{\varepsilon}] \) with respect to \((\mu, \omega)\), where \( \hat{\varepsilon} = Y - X\hat{\theta} \). In sum, we extend Baltagi et al. (2007) by enabling maximum-likelihood estimation and inference of dynamic spatiotemporal models.
EMPIRICAL ANALYSIS

This section describes the data, estimation results, and allocation insights. We also perform robustness checks and cross-validation to ensure the validity of findings.

Data

The German cosmetics market, about € 5.4 billion in size, consists of three equal sized segments: decorative cosmetics, face care, and body care (Nielsen 2008). We focus on the decorative cosmetics segment and analyze the proprietary data from the dominant market leader, whose identity remains confidential. The participating company operates in all seven Nielsen regions of Germany shown in Panel A of Figure 2. A Nielsen region aggregates areas that exhibit substantial similarity on various metrics (e.g., consumption patterns, demographics, psychographics, and purchasing power) and differs sufficiently with other Nielsen regions. Consequently, a Nielsen region need not adhere to political or provincial boundaries; for example, region 6 combines Saxony and Thuringia. Brand sales and advertising spending, respectively, were 200 million units and € 214 million over 29 months. On a monthly basis, the coefficient of variation (i.e., ratio of the standard deviation to the mean) for brand sales is 14% and for advertising is 45%, whereas market shares and prices fluctuate by less than 1% and 3%, respectively. Given the lack of variation in shares and prices over time, we focus on sales and advertising. National advertising (e.g., television, magazines, and national newspapers) consumes over 90% of the total budget; the rest, allocated to the seven regions, is spent on direct mail, local newspapers and radio. Although the regional ad spends are small, the coefficient of variation is more than 100% compared to 45% for the national spends. Table 2 summarizes the descriptive statistics.
Estimation Results

Robustness Checks. To account for endogeneity of advertising, we apply instrumental variables approach (e.g., Bronnenberg and Mahajan 2001, p. 286). At the regional level, we predict each region’s ad spends using spending in non-contiguous regions (i.e., neighbors of neighbors). At the national level, we predict national ad spends using two-period lagged national advertising. Because of the two-period spatial and temporal lags, we mitigate the correlations between the instruments and the spatial and serial error components (which have one-period dependence). In the Web Appendix, via Engle-Hendry-Richard test, we present evidence that the resulting instruments exhibit not only high goodness of fit, but also weak exogeneity. In words, weak exogeneity means that, when we factorize a joint density of sales and advertising, \( g(S, R, N) \), into the conditional density of sales given advertising \( g(S | R, N) \) and the marginal density of advertising \( g(R, N) \), the precise specification of the marginal density is not relevant and the model estimation using only the conditional density entails no loss of information.

In addition, we test for parameter constancy\(^1\) via Cusum test developed by Brown, Durbin and Evans (1975) and extended to dynamic models by Ploberger and Kramer (1992). If true parameters vary over time but the proposed model assumes constancy, then the model fit worsens and its residuals become large over time. Hence, the cumulative sum of residuals meanders away from mean zero and crosses the confidence bounds. The Cusum test results for our data indicate that the cumulative residuals lie within the confidence bounds, providing empirical support for parametric invariance. Given weak exogeneity and parametric invariance, according to Ericson and Irons (1994), these instruments are super-exogenous (see p. 14, ibid).

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\(^1\) We thank an anonymous reviewer for this suggestion.
Next, we compare the proposed model to other specifications, namely, the S-shaped and log advertising response model as well as the log-log model. We find that it outperforms the S-shaped model on the bias-corrected Akaike information criterion (AICc). The log advertising response model and the log-log model resulted in negative carryover rates and negative regional advertising effects, respectively, hence we reject these specifications as they lack face validity. We retain the proposed model because it enjoys stronger empirical support. We also compare the proposed model to one that includes intercepts. The results indicate that the proposed model performs better than the model with intercepts on both the information criterion (AICc = 28,342 for the proposed model versus 28,357 for the model with intercepts) and the likelihood ratio test (test statistic = 5.36 does not exceed the critical value of $\chi^2 = 14.07$), thereby rejecting the need for intercepts.

Finally, to verify whether regional and national spends are substitutes, we include advertising substitution effects in the sales model in Equation (1). We study two types of substitution effects: pure substitution and nested substitution. We model pure substitution effects by replacing $\beta_i \sqrt{R_i} + \alpha_i \sqrt{N_i}$ in Equation (1) with a single advertising variable $\beta_i \sqrt{R_i + N_i}$ for each region and then estimate this specification. The resulting AICc value of 28,350 exceeds the AICc value of 28,342 for the proposed model. Because this difference exceeds 2 points (Burnham and Anderson 2002, p. 70), the proposed model enjoys stronger empirical support. We test nested substitution by extending Equation (1) with the term $\psi_i \sqrt{R_i N_i}$ for each region. This specification nests the possibility of regional and national advertising being complements ($\psi_i > 0$), substitutes ($\psi_i < 0$) or independent ($\psi_i = 0$)—see Ingene and Parry (1995, p. 1195). We

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2 We thank an anonymous reviewer for this suggestion.
estimate this extended model and obtain an AICc value of 28,359, which is higher than that for the proposed model. Hence, the proposed model has stronger empirical support (also see the Web Appendix for insignificant values of \( \hat{\psi}_i \approx 0 \)).

\[\text{[Insert Table 3 about here]}\]

Model Fit and Forecasts. Table 3 shows that the system-wide \( R^2 \) is about 98%, indicating a good model fit to the in-sample data. To assess out-of-sample forecasting performance, we conduct cross-validation by fitting the model using the first twenty months of data and using the last nine months as the holdout sample. For each of the seven regions and the national level, the cross-validation \( R^2 \) vary from 97.12% to 98.06%, with the median of 97.84%. Hence, both the in-sample model fit and out-of-sample forecasts are satisfactory.

Sales Dynamics, Neighborhood Effects, and Spatial Heterogeneity. Table 3 further shows that the estimated carryover effects and advertising effectiveness vary across the seven regions. We note that all the signs for the estimated effects are positive, as it should be. The robust t-values indicate that, in each region, both ad spending and lagged sales have significant effects.

The estimated neighborhood effect \( \hat{\gamma} \) equals 0.1065. To test its significance jointly with all other parameters, we conduct the likelihood-ratio test. The test statistic of 32.02 exceeds the critical value \( \chi^2 \approx 3.84 \), thus favoring the inclusion of neighborhood effects in the model. This finding comports with the corrected Akaike information criterion. Substantively, the neighboring regions exert a positive impact on focal region’s sales. On average, 10.65% of the composite sales from neighboring regions spills over into the focal region. The elasticity of the various neighborhood effects equal \((0.0781, 0.0793, 0.1054, 0.1214, 0.0725, 0.2811, 0.1309)'\), which lends support to the presence of spatial heterogeneity. The large elasticities arise in regions with high
economic integration and business orientation. For example, region 6 is commercially integrated with both Bavaria to its southwest and Berlin to its north.

Using the coefficient of variation (CV), we assess the extent of spatial heterogeneity. We find that the CV for carryover effects is about 11%. Consequently, spatial heterogeneity for the carryover effects exists, but its magnitude is small. In contrast, the CVs for regional and national advertising are 39.6% and 37.1%, respectively, revealing a large spatial heterogeneity in advertising effectiveness. That is, as noted before, market data lends support to the presence of spatial heterogeneity. Across Germany, regional differences in media usage and advertising effectiveness reflect the variation in these estimates. Additionally, columns 5 and 6 of Table 3 provide the regional and national advertising elasticities, which show the CV for regional ad elasticities is 19.2% and that for national ad elasticities is 9.1%. This substantial variation in regional ad elasticities emphasizes the importance of spatial heterogeneity. Finally, we observe in Table 3 that the regional elasticities are an order of magnitude smaller than the national elasticities. Theoretically, this finding follows from the fact that the ratio of national to regional elasticities is proportional to the number of regions (under symmetric regions). Empirically, national advertising achieves higher reach compared to regional advertising, which the firm employs to enrich its media mix.

**Spatial and Serial Dependence.** The spatial correlation $\hat{\mu}$ is positive and significant at the 95% confidence level. Specifically, the maximum-likelihood estimate of $\hat{\mu} = 0.0403$ (standard error (se) = 0.019). Given this small but significant magnitude of $\hat{\mu}$, we test its stability using alternative contiguity matrices. We find $\hat{\mu} = 0.0400$ (se = 0.019) when the contiguity matrix is based on relative mean age across regions; $\hat{\mu} = 0.0402$ (se = 0.019) when it is defined on relative female to male ratio; $\hat{\mu} = 0.0464$ (se = 0.026) when it is defined on relative population
density. These results enhance our confidence in the finding of positive spatial dependence, which means positive (negative) shocks within a region increase (decrease) the sales of immediate neighbors. Because lower values of spatial dependence imply greater regional heterogeneity (Ataman, Mela, and Heerde 2007, p. 19), a small value of $\hat{\mu}$ emerges partly due to Nielsen’s way of combining regions that enhances inter-region heterogeneity.

The serial dependence $\hat{\phi}$ is negative and significant at the 95% confidence level. Specifically, the maximum-likelihood estimate of $\hat{\phi} = -0.3004$ (se = 0.0441). This finding reveals that, in addition to the spatial effects on the neighboring regions, shocks within a region alternate (in sign) over subsequent periods. A few plausible reasons include stocking behavior of consumers and retailers (Hall 1988; McGuire 1977), lag structure (Rao 1986), or inventory management decisions of distributors (Baganha and Cohen 1998; Ramey 1991). Given the magnitude of $\hat{\phi}$ is about 1/3, such oscillatory shocks lasts for three and half months because about a third of it dissipates every month. Based on these estimation results, we next address the substantive questions: how much to spend optimally on advertising, how much of it should be set aside for national advertising, and how to optimally allocate the rest to the seven Nielsen regions of Germany?

[Insert Table 4 about here]

Allocation Insights

*BDI vs. Optimal vs. Actual Allocations.* As illustrated in the example in literature review, we compute the BDI scores for the seven regions and the resulting BDI-based allocations — see Table 4 (Panel A). We recall the two drawbacks of the BDI approach: (1) the national budget cannot be determined, and (2) the optimality of these allocations cannot be ascertained due to its model-free nature. To overcome the second drawback, we use the fitted sales model, thus
extending the standard BDI approach. Equation (1) provides the sales model, which suggests that long-term sales equals \( \tilde{S} = D^{-1} \tilde{A} \), where \( \tilde{S} = (\tilde{S}_1, \ldots, \tilde{S}_K)' \), \( \tilde{A} = (\tilde{A}_i, \ldots, \tilde{A}_k)' \),

\[
\tilde{A}_i = (\beta_i \sqrt{R_i} + \alpha_i \sqrt{N_i}) \quad \text{and} \quad D = \begin{bmatrix}
\delta_1 & -\gamma_{12} & \cdots & -\gamma_{1K} \\
-\gamma_{12} & \delta_2 & \cdots & -\gamma_{2K} \\
\vdots & \vdots & \ddots & \vdots \\
-\gamma_{1K} & -\gamma_{2K} & \cdots & \delta_K 
\end{bmatrix}.
\]

Using the parameter estimates from Table 3 and margin information from Table 4 (see column 5 in Panel A), we then compute the regional long-term sales (see column 4) and revenues (see column 6). But, because the national spending is undeterminable (due to the first drawback), we have to use the actual national spending. Then, the resulting annual BDI profit, which equals total revenues minus the overall ad spends, is €70,088,902 (see Panel A of Table 4) — an increase of 0.37% over the actual profit.

Can managers achieve a higher profit? To this end, in Table 4, we compute the normative ad spends via Equations (7) and (8). We obtain the long-term sales \( S^* = D^{-1} A^* \), where

\[
S^* = (S_1^*, \ldots, S_K^*)', \quad A^* = (A_i^*, \ldots, A_k^*)' \quad \text{and} \quad A_i^* = (\beta_i \sqrt{R_i^*} + \alpha_i \sqrt{N_i^*}).
\]

The corresponding revenues are given in Panel B (see the columns 3 and 4). Then, the resulting annual optimal profit is €73,375,032, representing 4.7% increase over the BDI approach.

How do both the approaches compare with managers’ actual decisions? Because the actual annual profits were € 69,833,592 (see Panel B of Table 4 for details), the BDI approach yields 0.37% profit increase. In contrast, by using the proposed method, first, managers can earn a higher profit (~5.07%) compared to their own actions. Second, they learn about the optimal overall budget, which the BDI approach does not inform. By knowing the optimal overall budget, they discover whether they are over- or under-spending, which is unknowable in the absence of a benchmark. Third, they ascertain the optimal split between national and regional spends. Specifically, the actual split was 92.4%, while the optimal split should be \( \hat{\phi}^* = 85.9\% \)
(see Equation 9 and Panel B of Table 4). Thus, the proposed method yields a larger profit and reveals budget misallocations, whereas BDI leads to meager profit improvement. Furthermore, we make profit comparisons with and without neighborhood effects and spatial dependence, revealing that profit increase reduces by 22.68% when these effects are ignored. Together, the results further emphasize the importance of spatial effects in marketing (e.g., Bronnenberg and Mela 2004; Bronnenberg and Mahajan 2001; Bell and Song 2007).

**Budget Misallocations.** Many large companies misallocate resources; for example, Corstjens and Merrihue (2003, p. 8) interviewed senior executives from twenty leading global companies and “found widespread frustration on the matter [of misallocations]. Many complained that determining where and how marketing budgets should be allocated … seemed virtually impossible.” Our analysis comports with their findings, and the participating company also seems to be misallocating advertising dollars. Specifically, Figure 3 reveals region-specific misallocations and profit consequences.

[Insert Figure 3 about here]

Figure 3 identifies overspending in Regions 6 and 7 (i.e., the dot is above the dash) and under-spending in Regions 2, 3, 4, and 5 (i.e., the dot is below the dash). It also informs the magnitudes of misallocations. For example, in regions 6 and 7, the company overspends by about 89.5%, and 82.8%, respectively, whereas in regions 2, 3, 4 and 5 the magnitudes of under-spending are 62.5%, 115.9%, 60.1% and 191.5%, respectively. Although managers seem to operate near optimal in region 1, the model shows advertising budget should be increased by 10.2%.

We emphasize an important fact that the change in allocations are proportional to neither the regional sales, nor per capita regional sales as the BDI approach suggests. For instance,
regions 1 and 2 have higher sales than regions 3 and 5, yet the optimal allocation procedure recommends larger increases for regions 3 and 5.

Besides region-specific knowledge, the figure also highlights the misallocation in the national budget (~29.7%) and consequently the total budget (~24.4%). In the absence of a method for optimal allocation, which provides the normative benchmarks, it is indeed “virtually impossible”—as Corstjens and Merrihue (2003) noted—to know which regions under- and overspend. Thus, Proposition 1 injects this diagnostic information into the decision-making process.

**Optimal Re-allocation.** To identify the candidate regions for re-allocation, we compute 95% confidence interval around the optimal ad spends and resulting profit (by using the distributions of the estimated model parameters). Figure 3 shows that the actual ad spends in regions 1 through 5 lie within the 95% intervals, whereas those nationally and in regions 6 and 7 exceed the confidence interval. Hence, the firm overspends in regions 6 and 7 and at the national level. If managers re-allocate budget such that they reduce ad spends in regions 6 and 7 and nationally to fall within the confidence interval, then the resulting profit increases between 1.7% to 24.1% (see Figure 3). By eliminating these misallocations but keeping the total budget unchanged, sales would increase by 1.01% and profit by 3.08%. This result reinforces the previous finding (e.g., Mantrala, Sinha, and Zoltners 1992) that the budget re-allocation begets larger benefits than the change in overall budget. At the optimal allocation (i.e., setting the total budget and its allocation as per Proposition 1), sales would decrease by about 12.14%, but profit increases by 5.07%. This profit increase not only differs statistically from zero, but also is larger than the profit the firm generates from its current actions.
In closing, we acknowledge that managers may forgo part of the profit to gain larger volumes, for example, to negotiate on price with retailers. Indeed, over-advertising to increase volumes further supports the view that managers do not operate optimally. Based on these results, our discussions with the participating company nudged them to re-think their allocations of ad budget across national and regional advertising.

DISCUSSION

We discuss testing and allocation under time-varying parameters, continuous versus discrete time models, the effects of discount rate, and the conditions for pulsing versus even spending.

Time Varying Parameters. Prior research shows that model parameters for ad effectiveness and carryover effect may vary over time (e.g., Bass, Bruce, Majumdar, and Murthi 2007; Bronnenberg, Mahajan, and Vanhonacker 2000; Jedidi, Mela, and Gupta 1999; Mela, Gupta, and Lehmann 1997; Naik, Mantrala, and Sawyer 1998; Winer 1979). However, the proposed model assumes constancy of parameters. Hence, we test this assumption in the Web Appendix using Cusum test. As mentioned, we find that the confidence bounds contain the cumulative residuals, providing empirical support for parametric invariance.

Nonetheless, data from other markets might exhibit time-varying parameters. Besides time, parameters may evolve due to multiple region-specific covariates $Z_i(t)$ (e.g., demographics, income, purchasing power). To accommodate such dynamic effects, let

$$\frac{d\beta_i}{dt} = p(Z_i(t), t), \quad \frac{d\alpha_i}{dt} = q(Z_i(t), t), \quad \frac{d\delta_i}{dt} = r(Z_i(t), t), \quad \text{and} \quad \frac{d\gamma_{ij}}{dt} = u(Z_i(t), Z_j(t), t)$$

where $p(\cdot), q(\cdot), r(\cdot)$

---

3 We thank an anonymous reviewer for this insight.

4 We thank an anonymous reviewer for suggesting this extension.
and \( u(\cdot) \) specify the process functions for parametric evolution (e.g., see Gatignon and Hanssens 1987). Consequently, we augment the state space to include all \((K+3)\) dynamics for each region:

\[
\begin{align*}
\frac{dS_i}{dt}, \frac{d\beta_i}{dt}, \frac{d\alpha_i}{dt}, \frac{d\delta_i}{dt}, \frac{d\gamma_{ij}}{dt}
\end{align*}
\]

Then, we solve the multi-state (dimension \(K \times (K+3)\)) and multi-control (dimension \(K+1\)) problem and generalize Proposition 1 to markets with dynamically evolving parameters.

**Proposition 2:** For time-varying parameters, the optimal regional and national advertising expenditures are as follows: \( R^*_i(t) = \frac{1}{4} (\beta_i(t) b_i(t))^2 \) for \( i = 1, \cdots, K \) and

\[
N^*(t) = \frac{1}{4} \left( \sum_{i=1}^{K} \alpha_i(t) b_i(t) \right)^2,
\]

where

\[
\begin{bmatrix}
    b_1(t) \\
    b_2(t) \\
    \vdots \\
    b_K(t)
\end{bmatrix} =
\begin{bmatrix}
    \rho + \delta_1(t) & -\gamma_{21}(t) & \cdots & -\gamma_{K1}(t) \\
    -\gamma_{12}(t) & \rho + \delta_2(t) & \cdots & -\gamma_{K2}(t) \\
    \vdots & \vdots & \ddots & \vdots \\
    -\gamma_{1K}(t) & -\gamma_{2K}(t) & \cdots & \rho + \delta_K(t)
\end{bmatrix}^{-1} \begin{bmatrix}
    m_1 \\
    m_2 \\
    \vdots \\
    m_K
\end{bmatrix}.
\]

The total budget \( B^*(t) = N^*(t) + \sum_{i=1}^{K} R^*_i(t) \) and its optimal split \( \phi^*(t) = N^*(t) / B^*(t) \).

**Proof:** See Web Appendix.

A remarkable property of the optimal spends \((R^*_i(t), N^*_i(t))\) is that they do not depend on the process functions \( p(\cdot), q(\cdot), r(\cdot), u(\cdot) \) —they rely only on the resulting parameter values \((\alpha_i(t), \beta_i(t), \delta_i(t), \gamma_{ij}(t))\). In other words, we gain an insight that the optimal allocations depend on the outcome of how much the parameters changed rather than the process of how the change occurred.

**Continuous versus Discrete Time Models.** We transform Equation (1) to continuous-time for a reason.\(^5\) In discrete-time models, the optimal decisions are solutions to stochastic difference equations; whereas in continuous-time models, we exploit the continuity of time to obtain non-
stochastic (i.e., deterministic) differential equations. Due to the deterministic nature, the latter is more likely to yield analytical solutions than the former. We further elaborate this issue in the Web Appendix.

**Effect of Discount Rate.** Discount rate measures managers’ impatience. As discount rate increases, managers become more impatient and present-oriented. Hence, brand building efforts shrink as managers becomes impatient. We empirically compute the total spending for various discount rates from 0% to 12%, which spans the actual range stated in the company’s annual reports (4.25% to 6.25%), we find that the total spending decreases by 0.56% for every 1% increase in the discount rate (see Figure 4). Finally, we observe that the actual spending exceeds the optimal spending under $\rho = 0$, reinforcing our findings that the firm overspends.

[Insert Figure 4 about here]

**Pulsing versus Even Spending.** The optimal allocations recommended by this approach yield even spending policies. Past research shows pulsing is near optimal when parameters vary over time (Naik, Mantrala, and Sawyer 1998) or the response function is S-shaped (e.g., Mahajan and Muller 1986). We conduct the Cusum test to rule out time-varying parameters (see Robustness Checks) and compute information criteria to reject the S-shaped response function. Because the conditions required for pulsing do not apply to our setting, the prescribed even spending policy is optimal. For situations when parameters vary over time, we derived Proposition 2 that furnishes time-varying optimal allocations.

**CONCLUSION**

Recent marketing research (e.g., Albuquerque, Bronnenberg, and Corbett 2007; Ataman, Mela, and van Heerde 2007; Bronnenberg, Dhar, and Dubé 2007a) advances the estimation of
spatial or spatiotemporal models in marketing context, but important managerial questions still remain open. Specifically, how much should companies spend on advertising, how much of it should be set aside for national versus regional advertising, should they allocate the regional dollars to support the different regions?

To provide systematic answers, we propose a spatiotemporal model of advertising that accounts for sales dynamics, neighborhood effects, spatial heterogeneity, and spatial and serial dependence. Because of spatial and serial dependence, a correlated multivariate Brownian motion drives the sales dynamics which, in turn, results in a second-order differential equation for the value function with multiple states (i.e., regional sales) and multiple controls (i.e., regional advertising expenditure). We solve this normative problem analytically (see Web Appendix) to derive closed-form expressions for the optimal total budget and its optimal regional allocations for constant parameters (see Proposition 1) and time-varying parameters (see Proposition 2). In addition, we develop a method for estimation and inference of the proposed model, thereby extending Baltagi et al. (2007).

Our empirical analysis furnishes evidence for the presence of dynamic effects, neighborhood effects, spatial heterogeneity, and spatial and serial dependencies. Both the carryover effects and the effectiveness of advertising are significant across all regions. Carryover effects vary across regions by 11%, and regional and national advertising effectiveness varies by about 39.6% and 37.1%, respectively. Neighborhood effects are positive and directly affect the optimal allocations. Spatial dependence is positive and indirectly affects the optimal allocations via the precision of parameter estimates. Serial dependence is negative, which reveals that oscillatory shocks emerge due to unobservable factors (e.g., stockpiling by consumers and retailers). All parameters are estimated efficiently (e.g., t-values range from 1.9 to 14.5). If
significance is hard to detect, then shrinkage methods such as the hierarchical Bayesian approach (Bass et al. 2007), or direct constraints (Naik and Tsai 2005), or the Lasso (Tibshirani 1996) can be applied.

Our normative analysis shows that managers misallocate resources at both the national and regional levels. The total monthly budget should be reduced from €7.9 million to €5.9 million, and its split to national versus regional advertising should be changed from 92.4% to 85.9%. In addition, we observe specific regional misallocations; for example, the regions 6 and 7 overspend by 89.5% and 82.8%, respectively, and the regions 3 and 5 under-spend by 115.9% and 191.5%, respectively. By reducing both overspending and misallocations, optimal reallocation would enhance profit by 5.07%. Thus, companies can make informed decisions by using the proposed method for optimal spatiotemporal allocation of advertising budgets across different regions.

We close by identifying an avenue for future research. Because ad spending is committed by contract with media companies several months in advance through the upfront market (see Raman and Naik 2004, p. 11; Tellis 1998, p. 351; Belch and Belch 2004, p. 358), firms cannot change their media schedules in response to competition in the short run. Likewise, competitors also are committed and lack the flexibility to change media plans in response to focal firm’s advertising. Accordingly, academic literature shows that competitors seldom respond (e.g., Steenkamp, Nijs, Hanssens, and Dekimpe 2005). Hence, competitive response is minimal in the short term. However, this phenomenon may differ over a longer time horizon, and so we encourage further research to understand the role of competition.
REFERENCES


Nielsen (2008), *Retail Management Service (RMS)*. Frankfurt, Germany.


Figure 1

SPATIAL BUDGETING AND ALLOCATION SETTING

Total Annual Advertising Budget $B$

$\phi \times B$ Budget Split $(1-\phi) \times B$

National Advertising Budget $N$

Regional Advertising Budgets $R_i$

\[
\begin{align*}
N & \quad R_1 & \quad R_2 & \quad R_3 & \quad R_4 & \quad R_5 & \quad R_6 & \quad R_7 & \quad B \\
87.5 & \quad 1.3 & \quad 1.9 & \quad 0.7 & \quad 0.7 & \quad 0.9 & \quad 0.5 & \quad 1.1 & \quad 94.6 \\
? & \quad 1.1 & \quad 1.0 & \quad 1.2 & \quad 1.1 & \quad 1.2 & \quad 0.7 & \quad 0.8 & \quad ?
\end{align*}
\]

All amounts in million Euros
Figure 2

SEVEN NIELSEN REGIONS AND THEIR CONTIGUITY MATRIX

Panel A

Panel B

\[
\begin{bmatrix}
\text{Regions} & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \\
2 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
3 & 1 & 1 & 0 & 1 & 1 & 0 & 1 \\
4 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
5 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\
6 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\
7 & 1 & 0 & 0 & 0 & 0 & 1 & 0
\end{bmatrix}
\]

\( \tilde{c} = \)
Figure 3

RE-ALLOCATION OF AD SPENDS AND PROFIT

Actual level normalized to unity.
Figure 4

**EFFECT OF DISCOUNT RATES ON TOTAL BUDGET**

![Bar chart showing the effect of discount rates on total budget. The chart indicates the following values for the actual and discounted budgets: Actual: 7.89 million €, 0%: 6.16 million €, 6%: 5.95 million €, 12%: 5.76 million €.](image)
Table 1

SELECTED SPATIOTEMPORAL STUDIES IN MARKETING

<table>
<thead>
<tr>
<th>Authors</th>
<th>Area</th>
<th>Spatial Units</th>
<th>Spatial Heterogeneity</th>
<th>Neighborhood Effects</th>
<th>Spatial Dependence</th>
<th>Serial Dependence</th>
<th>Sales Dynamics</th>
<th>Marketing Decision Variables</th>
<th>Optimal Decisions</th>
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<tbody>
<tr>
<td>Gatignon, Eliashberg, and Roberts (1989)</td>
<td>Diffusion (international)</td>
<td>International (14 Countries)</td>
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<td>Diffusion (international)</td>
<td>International (56 Countries)</td>
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<td>Yes</td>
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<tr>
<td>Bell and Song (2007)</td>
<td>Diffusion</td>
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<td>Yes</td>
<td>Yes</td>
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<tr>
<td>Choi, Hui, and Bell (2010)</td>
<td>Diffusion</td>
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<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
<td>Numerical</td>
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<td>Demand Modeling</td>
<td>National (64 US IRI markets)</td>
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<td>Yes</td>
<td>Yes</td>
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<td>Ter Hofsteede, Wedel, and Steenkamp (2002)</td>
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<td>International (120 European Regions)</td>
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<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
<td>Positioning/ Store Image</td>
<td></td>
</tr>
<tr>
<td>Junk and Kannan (2006)</td>
<td>Targeting</td>
<td>National (160 consumers across US zip codes)</td>
<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
<td>Price</td>
<td></td>
</tr>
<tr>
<td>Junk and Kannan (2005)</td>
<td>Targeting &amp; Pricing</td>
<td>National (160 consumers across US zip codes)</td>
<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
<td>Price</td>
<td></td>
</tr>
<tr>
<td>Greenblatt (1981)</td>
<td>Pricing</td>
<td>International (3 Countries)</td>
<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
<td>Price explained</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Case (1991)</td>
<td>Pricing</td>
<td>Regional (8 Bed districts)</td>
<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
<td>Price</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Chau, Padmanabhan, and Sethuraman (2007)</td>
<td>Pricing</td>
<td>Local (1,550 continuous grid points in Singapore)</td>
<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
<td>Price</td>
<td>Numerical</td>
<td></td>
</tr>
<tr>
<td>Thomsen (2007)</td>
<td>Pricing</td>
<td>Local (US county; Simulation)</td>
<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
<td>Price</td>
<td>Promotion</td>
<td>Adv</td>
</tr>
<tr>
<td>Bronnenberg and Mahajan (2001)</td>
<td>Pricing &amp; Promotion</td>
<td>National (64 US IRI markets)</td>
<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
<td>Price</td>
<td>Promotion</td>
<td>Adv</td>
</tr>
<tr>
<td>Bhargava and Dondi (1999)</td>
<td>Advertising</td>
<td>Local (US city districts)</td>
<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
<td>Advertising</td>
<td>Analytical optimal allocation for the total and regional budgets</td>
</tr>
</tbody>
</table>
Table 2

DESCRIPTIVE STATISTICS

<table>
<thead>
<tr>
<th>Regions</th>
<th>Units Sold per Month</th>
<th>Ad Spends per Month €</th>
</tr>
</thead>
<tbody>
<tr>
<td>i</td>
<td>Average</td>
<td>Standard Deviation</td>
</tr>
<tr>
<td>1</td>
<td>1,250,554</td>
<td>168,392</td>
</tr>
<tr>
<td>2</td>
<td>1,555,474</td>
<td>216,659</td>
</tr>
<tr>
<td>3</td>
<td>1,083,338</td>
<td>146,819</td>
</tr>
<tr>
<td>4</td>
<td>996,878</td>
<td>138,016</td>
</tr>
<tr>
<td>5</td>
<td>1,204,748</td>
<td>163,440</td>
</tr>
<tr>
<td>6</td>
<td>394,853</td>
<td>53,768</td>
</tr>
<tr>
<td>7</td>
<td>663,744</td>
<td>93,272</td>
</tr>
<tr>
<td>National</td>
<td>7,149,589</td>
<td>968,735</td>
</tr>
</tbody>
</table>

Sample Size T = 29 months
Table 3

PARAMETER ESTIMATES

<table>
<thead>
<tr>
<th>Region</th>
<th>Lagged Sales</th>
<th>Regional Advertising</th>
<th>National Advertising</th>
<th>Regional Elasticity</th>
<th>National Elasticity</th>
</tr>
</thead>
<tbody>
<tr>
<td>i</td>
<td>( \hat{\lambda}_i )</td>
<td>( \hat{\beta}_i )</td>
<td>( \hat{\alpha}_i )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>0.6384</td>
<td>109.01</td>
<td>135.82</td>
<td>0.01</td>
<td>0.14</td>
</tr>
<tr>
<td></td>
<td>(14.20)</td>
<td>(2.58)</td>
<td>(5.30)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>0.6570</td>
<td>143.31</td>
<td>152.13</td>
<td>0.02</td>
<td>0.13</td>
</tr>
<tr>
<td></td>
<td>(14.56)</td>
<td>(2.89)</td>
<td>(4.61)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>0.5702</td>
<td>136.17</td>
<td>134.83</td>
<td>0.01</td>
<td>0.16</td>
</tr>
<tr>
<td></td>
<td>(10.77)</td>
<td>(1.90)</td>
<td>(6.23)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>0.5720</td>
<td>123.54</td>
<td>116.70</td>
<td>0.01</td>
<td>0.15</td>
</tr>
<tr>
<td></td>
<td>(9.70)</td>
<td>(2.25)</td>
<td>(5.57)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>0.6080</td>
<td>157.77</td>
<td>145.34</td>
<td>0.02</td>
<td>0.16</td>
</tr>
<tr>
<td></td>
<td>(14.00)</td>
<td>(2.56)</td>
<td>(6.08)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>0.4617</td>
<td>41.01</td>
<td>39.52</td>
<td>0.01</td>
<td>0.13</td>
</tr>
<tr>
<td></td>
<td>(3.10)</td>
<td>(1.97)</td>
<td>(4.95)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>0.5878</td>
<td>61.02</td>
<td>71.83</td>
<td>0.01</td>
<td>0.14</td>
</tr>
<tr>
<td></td>
<td>(9.31)</td>
<td>(3.09)</td>
<td>(4.98)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Neighborhood Effect \( \hat{\gamma} \) 0.1065 (se = 0.065)
Spatial Dependence \( \hat{\mu} \) 0.0403 (se = 0.019)
Serial Dependence \( \hat{\omega} \) -0.3004 (se = 0.044)
System-wide R² 98.45%

(t-values in parentheses); se = standard error
Table 4

MONTHLY ALLOCATIONS AND ANNUAL PROFIT

Panel A: BDI-Based Results

<table>
<thead>
<tr>
<th>Region</th>
<th>BDI</th>
<th>BDI Allocations*</th>
<th>Long Term Sales</th>
<th>Margin</th>
<th>Revenues</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\tilde{R}_i$</td>
<td>$\tilde{S}_i$</td>
<td>$m_i$</td>
<td>$m_i \times \tilde{S}_i$</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>109</td>
<td>94,847</td>
<td>1,415,325</td>
<td>1.83</td>
<td>2,590,045</td>
</tr>
<tr>
<td>2</td>
<td>99</td>
<td>86,636</td>
<td>1,734,002</td>
<td>1.81</td>
<td>3,138,544</td>
</tr>
<tr>
<td>3</td>
<td>112</td>
<td>97,447</td>
<td>1,250,012</td>
<td>1.76</td>
<td>2,200,021</td>
</tr>
<tr>
<td>4</td>
<td>107</td>
<td>93,052</td>
<td>1,151,795</td>
<td>1.61</td>
<td>1,854,390</td>
</tr>
<tr>
<td>5</td>
<td>111</td>
<td>96,679</td>
<td>1,384,851</td>
<td>1.81</td>
<td>2,506,580</td>
</tr>
<tr>
<td>6</td>
<td>70</td>
<td>60,742</td>
<td>454,395</td>
<td>1.18</td>
<td>536,186</td>
</tr>
<tr>
<td>7</td>
<td>76</td>
<td>66,231</td>
<td>750,120</td>
<td>1.20</td>
<td>900,144</td>
</tr>
<tr>
<td>National</td>
<td>—</td>
<td>Cannot Be Computed</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

National Ad Spend
(From Data)

| Totals (Monthly) | 7,885,168 |
| Annual BDI Profits | €70,088,902 |

* $\tilde{R}_i \propto \frac{BDI}{100} \times$ Total Regional Budget

Panel B: Optimal and Actual Results

<table>
<thead>
<tr>
<th>Region</th>
<th>Optimal Allocations(^{(1)})</th>
<th>Optimal Long Term Sales</th>
<th>Optimal Revenues</th>
<th>Actual Allocations(^{(2)})</th>
<th>Actual Long Term Sales</th>
<th>Actual Revenues</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$R^<em>_i, N^</em>_i$</td>
<td>$S^*_i$</td>
<td>$m_i \times S^*_i$</td>
<td>$R_i, N_i$</td>
<td>$S_i^*$</td>
<td>$m_i \times S_i^*$</td>
</tr>
<tr>
<td>1</td>
<td>122,778</td>
<td>1,225,541</td>
<td>2,242,740</td>
<td>111,399</td>
<td>1,424,366</td>
<td>2,606,590</td>
</tr>
<tr>
<td>2</td>
<td>253,082</td>
<td>1,573,827</td>
<td>2,848,627</td>
<td>155,679</td>
<td>1,774,272</td>
<td>3,211,432</td>
</tr>
<tr>
<td>3</td>
<td>134,541</td>
<td>1,094,709</td>
<td>1,926,689</td>
<td>62,297</td>
<td>1,230,650</td>
<td>2,165,944</td>
</tr>
<tr>
<td>4</td>
<td>98,682</td>
<td>998,805</td>
<td>1,608,076</td>
<td>61,614</td>
<td>1,130,397</td>
<td>1,819,940</td>
</tr>
<tr>
<td>5</td>
<td>209,225</td>
<td>1,246,952</td>
<td>2,256,983</td>
<td>71,756</td>
<td>1,364,032</td>
<td>2,468,897</td>
</tr>
<tr>
<td>6</td>
<td>4,483</td>
<td>378,307</td>
<td>446,402</td>
<td>43,008</td>
<td>456,625</td>
<td>538,818</td>
</tr>
<tr>
<td>7</td>
<td>15,460</td>
<td>619,922</td>
<td>743,906</td>
<td>89,882</td>
<td>744,178</td>
<td>893,014</td>
</tr>
<tr>
<td>National</td>
<td>5,120,586</td>
<td>12,073,423</td>
<td>7,289,534</td>
<td>7,885,169</td>
<td>8,054,711</td>
<td>13,704,635</td>
</tr>
</tbody>
</table>

Annual Profits

| €73,375,032 | €69,833,592 |

Split

| 85.93% | 92.44% |

\(^{(1)}\) Based on Proposition 1, parameter values in Table 3, and margins in Panel A of Table 4.

\(^{(2)}\) Average spending based on 28 months
WEB APPENDIX

PROOF OF PROPOSITION 1 (CONSTANT PARAMETERS)

To derive optimal budget and allocation, we solve the HJB equation (6)
\[ \rho V = \text{Max}[\sum_i m_i S_i - \sum_i R_i - N + \sum_i \dot{V}_i f_i + 0.5 \text{Tr}(\dot{V} \Sigma_v)]. \]  
(A-1)

Differentiating the right hand side of (A-1) with respect to \( R_i \), we get the first-order conditions (FOCs) for each region \( i \):
\[ -1 + (\beta \dot{V}_i \cdot \frac{1}{2 \sqrt{R_i}} = 0, \]  
(A-2)

which upon re-arrangement yields the implicit optimal ad expenditures for each region \( i \):
\[ R_i^* = \frac{1}{4} (\beta \dot{V}_i)^2. \]  
(A-3)

Similarly, when we differentiate the right hand side of (A-1) with respect to \( N \), we get the FOC for the national level,
\[ -1 + \sum_i \alpha \dot{V}_i \frac{1}{2 \sqrt{N}} = 0, \]  
(A-4)

which upon re-arrangement yields the implicit optimal national ad expenditure \( N^* \):
\[ N^* = \frac{1}{4} (\sum_i \alpha \dot{V}_i)^2. \]  
(A-5)

Then, we observe from (A-3) and (A-5) that \((R_i^*, N^*)\) are implicit functions of \( \dot{V}_i \). To explicitly express \( \dot{V}_i \) in terms of model parameters, we conjecture and then confirm that the value function is
\[ V(S_1, \cdots, S_K) = a + \sum_{i=1}^K b_i S_i. \]  
(A-6)

From (A-6), we find that \( \dot{V}_i = b_i \) for \( i = 1, \cdots, K, \) and \( \dot{V} = 0 \). Substituting \( b_i \) for \( \dot{V}_i \) in (A-3) and (A-5), we obtain
\[ R_i^* = \frac{1}{4} (\beta b_i)^2 \quad \text{and} \quad N^* = \frac{1}{4} (\sum_{i=1}^K \alpha b_i)^2. \]  
(A-7)

Substituting (A-6), \( \dot{V}_i = b_i \) and \( \dot{V} = 0 \) in (A-1), we obtain
\[ \rho(a + \sum_{i=1}^K b_i S_i) = \sum_i m_i S_i - \sum_i R_i^* - N^* + \sum_{i=1}^K b_i (-\delta_i S_i + \beta_i \sqrt{R_i^*} + \alpha_i \sqrt{N^*} + \sum_{j=i}^{K} \gamma_{ij} S_j). \]  
(A-8)

By equating the coefficients of \( S_i \) on both the sides of (A-8), we find that \( b_i \) can be obtained from
\[
\begin{bmatrix}
  b_1 \\
  b_2 \\
  \vdots \\
  b_K
\end{bmatrix}
= \begin{bmatrix}
  \rho + \delta_1 & -\gamma_{21} & \cdots & -\gamma_{1K} \\
  -\gamma_{21} & \rho + \delta_2 & \cdots & -\gamma_{2K} \\
  \vdots & \vdots & \ddots & \vdots \\
  -\gamma_{K1} & -\gamma_{K2} & \cdots & \rho + \delta_K
\end{bmatrix}
\begin{bmatrix}
  m_1 \\
  m_2 \\
  \vdots \\
  m_K
\end{bmatrix}
\]  
for all \( i \). Substituting \( b_i \) in (A-7), we obtain explicitly
the optimal regional advertising for all the regions:

44
and the optimal national advertising:

\[ N^* = \frac{1}{4} (\alpha_1 b_1 + \cdots + \alpha_K b_K)^2. \]  

(A-10)

The closed-form expressions in (A-9) and (A-10) provide the dynamically optimal spatiotemporal allocations of advertising budgets across all regions and at the national level, thus proving Proposition 1.

**Proof of Proposition 2 (Time-varying Parameters)**

Let parameters evolve according to the differential equations

\[
\frac{d\alpha_i}{dt} = q(Z_i(t), t), \quad \frac{d\delta_i}{dt} = r(Z_i(t), t), \quad \text{and} \quad \frac{d\gamma_{ij}}{dt} = u(Z_i(t), Z_j(t), t). 
\]

Then we augment the state space to incorporate the dynamics of parameters, which modify the HJB equation in (A-1):

\[
\rho V = \max \left[ \left( \sum_i m_i S_i - \sum_i R_i - N \right) + \sum_i \dot{V}_{S_i} p_i + \sum_i \dot{V}_{\beta_i} \beta_i + \sum_i \dot{V}_{\alpha_i} \alpha_i + \sum_i \dot{V}_{\delta_i} \delta_i + \sum_i \dot{V}_{\gamma_{ij}} \gamma_{ij} + 0.5 \text{Tr}(\dot{V} \Sigma_z) \right],
\]

(A-11)

where \( \dot{V}_{S_i} = \partial V / \partial S_i, \dot{V}_{\beta_i} = \partial V / \partial \beta_i, \dot{V}_{\alpha_i} = \partial V / \partial \alpha_i, \dot{V}_{\delta_i} = \partial V / \partial \delta_i, \) and \( \dot{V}_{\gamma_{ij}} = \partial V / \partial \gamma_{ij}. \) We then obtain the first-order conditions (FOCs):

\[
R_i^* = \frac{1}{4} (\beta_i(t) \dot{V}_{S_i})^2, \quad \text{and} \quad N^* = \frac{1}{4} (\sum_i \alpha_i(t) \dot{V}_{S_i})^2, 
\]

(A-12)

and observe that \( (R_i^*, N^*) \) are implicit functions of \( \dot{V}_{S_i}. \) To express \( \dot{V}_{S_i} \) in terms of model parameters, we augment (A-6) as follows:

\[
V(S_1, \ldots, S_K, \beta_1, \ldots, \beta_K, \alpha_1, \ldots, \alpha_K, \delta_1, \ldots, \delta_K, \gamma_{12}, \ldots, \gamma_{KK-1}) = a + \sum_{i=1}^{K} b_i S_i + \sum_{i=1}^{K} c_i \beta_i + \sum_{i=1}^{K} d_i \alpha_i + \sum_{i=1}^{K} e_i \delta_i + \sum_{j=1}^{K} \sum_{\mu=1}^{K} g_{ij} \gamma_{ij}
\]

(A-13)

Because \( \dot{V}_{S_i} = b_i \) and \( \dot{V} = 0 \) from (A-13), we substitute \( b_i \) for \( \dot{V}_{S_i} \) in (A-12) to get

\[
R_i^*(t) = \frac{1}{4} (\beta_i(t) b_i(t))^2 \quad \text{and} \quad N^*(t) = \frac{1}{4} (\sum_i \alpha_i(t) b_i(t))^2. 
\]

(A-14)

Substituting (A-13), \( \dot{V}_{S_i} = b_i \) and \( \dot{V} = 0 \) in (A-11), and following the same steps described in the proof of Proposition 1, we find that

\[
\begin{bmatrix}
  b_1(t) \\
  b_2(t) \\
  \vdots \\
  b_K(t)
\end{bmatrix} =
\begin{bmatrix}
  \rho + \delta_1(t) & -\gamma_{12}(t) & \cdots & -\gamma_{1K}(t) \\
  -\gamma_{21}(t) & \rho + \delta_2(t) & \cdots & -\gamma_{2K}(t) \\
  \vdots & \vdots & \ddots & \vdots \\
  -\gamma_{K1}(t) & -\gamma_{K2}(t) & \cdots & \rho + \delta_K(t)
\end{bmatrix}^{-1}
\begin{bmatrix}
  m_1 \\
  m_2 \\
  \vdots \\
  m_K
\end{bmatrix},
\]

furnishing the optimal regional and national advertising strategies:

\[
R_i^*(t) = \frac{1}{4} (\beta_i(t) b_i(t))^2 \quad \text{and} \quad N^*(t) = \frac{1}{4} (\sum_i \alpha_i(t) b_i(t) + \cdots + \alpha_K(t) b_K(t))^2 
\]

(A-15)

**Derivation of the Error Covariance Matrix \( \tilde{\Sigma}_e(\mu, \omega) \).**
To derive the error covariance matrix $\text{Cov}(\varepsilon) = \Sigma_\varepsilon$, we express equation (2) in the vector form $\varepsilon_i = \mu C \varepsilon_i + \eta_i$, so that $B \varepsilon_i = \eta_i$, where $B = I - \mu C$. We then stack the error terms such that all $(T-1)$ observations of each region $i$ are kept together and then stacked by regions (as in equation 11). That is, the vector $\eta = (\eta_1', \ldots, \eta_K')'$, where $\eta_i = (\eta_{i,1}, \ldots, \eta_{i,T})'$, is a column vector of dimension $(T-1)K$. We achieve the same arrangement for $B \varepsilon_i$ via $(B \otimes I_{T-1}) \varepsilon_i$. Consequently, $\eta = (B \otimes I_{T-1}) \varepsilon_i$. This step recovers the spatially uncorrelated error $\eta$. To remove the serial dependence from $\eta$, we construct the matrix $M(\omega) = \frac{1}{(1 - \omega^2)} \begin{bmatrix} 1 & \omega & \omega^2 & \cdots & \omega^{T-2} \\ \omega & 1 & \omega & \cdots & \omega^{T-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \omega^{T-2} & \omega^{T-3} & \omega^{T-4} & \cdots & 1 \end{bmatrix}$, which captures the influence of serial dependence over time. Then we invert the Cholesky factor of $M$ to obtain $L_i$; i.e. $L_i = L_i$. Pre-multiplying $\eta$ by $(I_K \otimes L_i) \eta_i = (I_K \otimes L_i)(B \otimes I_{T-1}) \varepsilon_i$. This step converts the serially correlated error $\eta$ into a set of classical disturbance $v_t$ of equation (3), which are free from spatial and serial dependencies. That is, $(I_K \otimes L_i) \eta = v_t$, where $v = (v_1', \ldots, v'_K)'$, and $v_t = (v_{T,1}, \ldots, v_{T,T})'$. Applying the result $(A \otimes B)(C \otimes D) = AC \otimes BD$, we see that $(I_K \otimes L_i)(B \otimes I_{T-1}) \varepsilon_i = (B \otimes L_i) \varepsilon_i$, which yields $\varepsilon = (B \otimes L_i)^{-1} v_t$. We thus find the covariance $\text{Cov}(\varepsilon) = \hat{\Sigma}_\varepsilon(\mu, \omega) = (B \otimes L_i)^{-1}(\Sigma_v \otimes I_{T-1})(B \otimes L_i)^{-1} \varepsilon$. Using $(A \otimes B)^{-1} = A^\dagger \otimes B^\dagger$ and $(A \otimes B)' = A' \otimes B'$, we simplify the error covariance matrix:

$$
\hat{\Sigma}_\varepsilon(\mu, \omega) = (B \otimes L_i)^{-1}(\Sigma_v \otimes I_{T-1})(B \otimes L_i)^{-1} \varepsilon \\
= (B^{-1} \otimes L_i^\dagger)(\Sigma_v \otimes I_{T-1})(B^{-1} \otimes L_i^\dagger) \\
= (B^{-1} \Sigma_v \otimes I_{T-1})(B^{-1} \otimes L_i^\dagger) \\
= (B^{-1} \Sigma_v B^{-1} \otimes L_i^\dagger L_i^\dagger) \\
= ((I - \mu C)^{-1} \Sigma_v (I - \mu C)^{-1} \otimes M(\omega)),
$$

which is shown in Equation (13).

**ROBUSTNESS CHECKS**

**Validity of Instruments.** Instruments are *weakly exogenous* if they are correlated with the independent variables and uncorrelated with the residuals. In addition, they are *super-exogenous* if model parameters are constant over time (Ericson and Irons 1994, p. 14). To ascertain their quality, we conduct three robustness checks: (i) the goodness of fit; (ii) independence between instruments and residuals; (iii) Cusum test for parameter constancy.

Table A presents the results. As for (i), we find high $R^2$ between the instruments and the actual regional and national advertising expenditures (see the first row in Table A).

**Table A: Validity of Instruments**

<table>
<thead>
<tr>
<th></th>
<th>Reg. 1</th>
<th>Reg. 2</th>
<th>Reg. 3</th>
<th>Reg. 4</th>
<th>Reg. 5</th>
<th>Reg. 6</th>
<th>Reg. 7</th>
<th>Nat.</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>R-squared</strong></td>
<td>0.9249</td>
<td>0.9389</td>
<td>0.9083</td>
<td>0.8851</td>
<td>0.9001</td>
<td>0.9085</td>
<td>0.9060</td>
<td>0.6442</td>
</tr>
</tbody>
</table>
t-values \((z_1 - z_2)^*\) * 
1.42 1.17 0.70 1.12 1.20 1.26 1.24 

\(t\)-values \((z_1 + z_2)^*\) * 
-0.22 -0.57 -0.98 -0.66 -0.52 -0.12 -0.24 

*critical \(t\)-value = 2.05

To test weak exogeneity (see Engle, Hendry, and Richard 1983, p. 290), we obtain the residuals from the conditional model (\(\hat{e}_u\) via equation 1) and the marginal models for regional advertising \(\hat{e}_u = (R_u - \overline{R}_u)\) and national advertising \(\hat{e}_u = (N_i - \overline{N}_i)\), where \(\overline{R}_u\) and \(\overline{N}_i\) are the averages over time. For each region, we then compute the correlations \(r_{12}^i = \text{corr}(\hat{e}_u, \hat{e}_u)\) and \(r_{13}^i = \text{corr}(\hat{e}_u, \hat{e}_u)\) to test the joint hypothesis \(H_0: r_{12}^i = r_{13}^i = 0\). Using Fisher’s \(z\)-transform \(z_1 = \tanh^{-1}(r_{12}^i)\) and \(z_2 = \tanh^{-1}(r_{13}^i)\), we test the significance of \((z_1 + z_2)\) and \((z_1 - z_2)\) relative to their variances given by \(\text{Var}(z_1) + \text{Var}(z_2) = 2\text{Cov}(z_1, z_2)\), respectively. Based on Meng, Rosenthal, and Rubin (1992, p. 175), we know \(\text{Var}(z_1) = \text{Var}(z_2) = (T - 3)^{-1}\) and

\[
\text{Cov}(z_1, z_2) = r_{23} + 0.5[r_{12}^i r_{13}^i (r_{13}^i - r_{23})^2 \left(1 - r_{12}^i \right)^2 (1 - r_{13}^i)^2 - r_{12}^i r_{13}^i],
\]

where \(r_{23}^i = \text{corr}(\hat{e}_u, \hat{e}_u)\). The resulting \(t\)-values (see the second and third rows of Table A) are smaller than the critical \(t\)-value = 2.05 at 28 degrees of freedom and 95% confidence level. Given the support for (i) and (ii), we thus establish weak exogeneity.

To detect stability of parameters over time, we compute the Cusum statistic based on Brown, Durbin, and Evans (1975) and Ploberger and Kramer (1992). Specifically, we use the first 20 data points to obtain the parameters estimates and use the last 9 months as the holdout period. Next, we estimate parameters recursively and calculate residuals recursively from the twenty first month onwards until \(T = 29\). If the Cusum statistic lies within the significance lines given by \(\pm \sqrt{T} + 2a\sqrt{T}\), where \(a = 0.845\) for 90% level of significance, then model parameters are considered as stable. The cumulative sum of residuals for each region always lies within the 95% bounds. Hence, our data comports with parametric constancy over time. Given the support for (iii), together with weak exogeneity, we thus establish super exogeneity.

**Alternative Model Specifications**

We compare the proposed model to other specifications: no neighborhood effects, no spatial heterogeneity, no spatial dependence, the S-shaped response \(e^x / (1 + e^x)\), model with intercepts, the log response \(\ln(x)\), and the Cobb-Douglas (log-log) model. Table B shows the results based on the bias-corrected Akaike Information Criteria (AICc), which balances fidelity (goodness of fit) and parsimony (fewer parameters).

<table>
<thead>
<tr>
<th>Model</th>
<th>Log-Likelihood</th>
<th>No. of Parameters</th>
<th>No. of Obs.</th>
<th>AICc</th>
</tr>
</thead>
<tbody>
<tr>
<td>Proposed</td>
<td>-14,044</td>
<td>24</td>
<td>196</td>
<td>28,342</td>
</tr>
<tr>
<td>No Neighborhood Effects</td>
<td>-14,060</td>
<td>23</td>
<td>196</td>
<td>28,371</td>
</tr>
<tr>
<td>No Spatial Heterogeneity</td>
<td>-15,057</td>
<td>6</td>
<td>196</td>
<td>30,325</td>
</tr>
<tr>
<td>No Spatial Dependency</td>
<td>-14,046</td>
<td>23</td>
<td>196</td>
<td>28,343</td>
</tr>
<tr>
<td>S-shaped</td>
<td>-14,054</td>
<td>24</td>
<td>196</td>
<td>28,362</td>
</tr>
</tbody>
</table>
The proposed model attains the lowest AIC\(_C\) amongst specifications with positive parameters estimates. Because the difference in AIC\(_C\) values between the proposed model and the other specifications exceeds 2, the proposed model enjoys stronger empirical support (Burnham and Anderson 2002, p. 70). Thus, we retain the proposed model for normative and empirical analyses.

Next, we provide the results of the model that nests advertising substitution effects into Equation (1); namely, we append Equation (1) with \(\psi_i \sqrt{R_iN_i}\) for each region. The resultant AIC\(_C\) value of 28,359 exceeds the AIC\(_C\) value of 28,342, indicating that the substitution effects are negligible. Indeed, the values of parameter \(\hat{\psi}_i \approx 0\) are insignificant (see Table C below).

### Table C: Substitution Effects

<table>
<thead>
<tr>
<th>Region</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\psi)</td>
<td>-0.0012</td>
<td>-0.0005</td>
<td>-0.0004</td>
<td>-0.0013</td>
<td>-0.0018</td>
<td>-0.0003</td>
<td>0.0004</td>
</tr>
<tr>
<td>t-values</td>
<td>-0.1211</td>
<td>-0.0471</td>
<td>-0.0331</td>
<td>-0.1098</td>
<td>-0.1395</td>
<td>-0.0425</td>
<td>0.0587</td>
</tr>
</tbody>
</table>

For further verification, we conduct a likelihood ratio (LR) test, which yields a value of 1.98 and it is less than the critical value of \(\chi^2 = 14.07\). Hence, substitution effects are negligible.

### Derivation of Continuous Time Sales Model

Below we provide a heuristic derivation of the continuous time model from the discrete time sales model in Equations (1), (2) and (3). We elucidate using two regions and then generalize for \(K\) regions. In matrix form, the two regions sales models become

\[
\begin{bmatrix}
S_{t1} \\
S_{t2}
\end{bmatrix}
= \begin{bmatrix}
\lambda_{1} & \gamma_{12} \\
\gamma_{21} & \lambda_{2}
\end{bmatrix}
\begin{bmatrix}
S_{t-11} \\
S_{t-12}
\end{bmatrix}
+ \begin{bmatrix}
\beta_{1} \sqrt{R_{t1}} & + \alpha_{1} \sqrt{N_{t}} \\
\beta_{2} \sqrt{R_{t2}} & + \alpha_{2} \sqrt{N_{t}}
\end{bmatrix}
\begin{bmatrix}
\epsilon_{t1} \\
\epsilon_{t2}
\end{bmatrix}
+ \begin{bmatrix}
\eta_{t1} \\
\eta_{t2}
\end{bmatrix},
\]  

(A-17)

\[
\begin{bmatrix}
\epsilon_{t1} \\
\epsilon_{t2}
\end{bmatrix}
= \mu
\begin{bmatrix}
0 & c_{12} \\
c_{21} & 0
\end{bmatrix}
\begin{bmatrix}
\epsilon_{t1} \\
\epsilon_{t2}
\end{bmatrix}
+ \begin{bmatrix}
\eta_{t1} \\
\eta_{t2}
\end{bmatrix},
\]  

(A-18)

\[
\begin{bmatrix}
\eta_{t1} \\
\eta_{t2}
\end{bmatrix}
= \omega
\begin{bmatrix}
0 & \eta_{t-1} \\
\eta_{t-1} & 0
\end{bmatrix}
\begin{bmatrix}
\eta_{t1} \\
\eta_{t2}
\end{bmatrix}
+ \begin{bmatrix}
\nu_{t1} \\
\nu_{t2}
\end{bmatrix},
\]  

(A-19)

where \((\epsilon_{t1}, \epsilon_{t2})' \sim N(0, \Sigma_{\epsilon})\) and \((\nu_{t1}, \nu_{t2})' \sim N(0, \Sigma_{\nu})\). We rewrite (A-18) as

\[
\begin{bmatrix}
\epsilon_{t1} \\
\epsilon_{t2}
\end{bmatrix}
= \left( I_{(2 \times 2)} - \mu
\begin{bmatrix}
0 & c_{12} \\
c_{21} & 0
\end{bmatrix}\right)^{-1}
\begin{bmatrix}
\eta_{t1} \\
\eta_{t2}
\end{bmatrix}
= B^{-1}
\begin{bmatrix}
\eta_{t1} \\
\eta_{t2}
\end{bmatrix}.
\]  

(A-20)

To obtain the continuous time version of (A-17), we transform it as follows:
Hence (A-20) becomes
\[
\begin{bmatrix}
\varepsilon_1 \\
\varepsilon_2
\end{bmatrix}
dt = B^{-1}
\begin{bmatrix}
\eta_1 \\
\eta_2
\end{bmatrix}
dt
\]
in continuous time. To obtain the continuous time version of (A-19), we transform it as follows:
\[
\begin{bmatrix}
\eta_{t} - \eta_{t-1} \\
\eta_{2t} - \eta_{2t-1}
\end{bmatrix}
= -I
\begin{bmatrix}
\phi & 0 \\
0 & \phi
\end{bmatrix}
\begin{bmatrix}
\eta_{t-1} \\
\eta_{2t-1}
\end{bmatrix} + \begin{bmatrix}
v_{t} \\
v_{2t}
\end{bmatrix},
\]
\[
\begin{bmatrix}
d\eta_{1} \\
d\eta_{2}
\end{bmatrix} = -\Omega
\begin{bmatrix}
\eta_{1} \\
\eta_{2}
\end{bmatrix}
dt + \begin{bmatrix}
v_{1} \\
v_{2}
\end{bmatrix}
dt,
\]
\[
\begin{bmatrix}
d\eta_{1} \\
d\eta_{2}
\end{bmatrix} = -\Omega
\begin{bmatrix}
\eta_{1} \\
\eta_{2}
\end{bmatrix}
dt + A
\begin{bmatrix}
dW_{1}(t) \\
dW_{2}(t)
\end{bmatrix},
\]
where \(d\eta_{t}\) is a time-transformed Wiener process, \(\Omega = I - \begin{bmatrix}
\phi & 0 \\
0 & \phi
\end{bmatrix}\), \(dW_{t}\) is the standard Wiener process, and \(AA' = \Sigma_{v}\). We rewrite this equation as follows
\[
\begin{bmatrix}
\eta_{1} \\
\eta_{2}
\end{bmatrix}
dt = \Omega^{-1}
\begin{bmatrix}
d\tilde{W}_{1}(t) \\
d\tilde{W}_{2}(t)
\end{bmatrix},
\]
which yields
\[
\begin{bmatrix}
\eta_{1} \\
\eta_{2}
\end{bmatrix}
dt = \Omega^{-1}
\begin{bmatrix}
d\tilde{W}_{1}(t) \\
d\tilde{W}_{2}(t)
\end{bmatrix},
\]
where \(d\tilde{W}_{t}\) follows a Gauss-Markov process. Then (A-20) becomes
\[
\begin{bmatrix}
\varepsilon_1 \\
\varepsilon_2
\end{bmatrix}
dt = B^{-1}\Omega^{-1}
\begin{bmatrix}
d\tilde{W}_{1}(t) \\
d\tilde{W}_{2}(t)
\end{bmatrix}.
\]
Substituting (A-23) in (A-21) yields
\[
\begin{bmatrix}
ds_{1} \\
ds_{2}
\end{bmatrix} = \begin{bmatrix}
-\delta_{1} \gamma_{12} \\
-\delta_{2} \gamma_{21}
\end{bmatrix}
\begin{bmatrix}
S_{1}(t) \\
S_{2}(t)
\end{bmatrix} + \begin{bmatrix}
\beta_{1} \sqrt{R_{1}(t)} + \alpha_{1} \sqrt{N(t)} \\
\beta_{2} \sqrt{R_{2}(t)} + \alpha_{2} \sqrt{N(t)}
\end{bmatrix}
dt + B^{-1}\Omega^{-1}
\begin{bmatrix}
d\tilde{W}_{1}(t) \\
d\tilde{W}_{2}(t)
\end{bmatrix}.
\]
Generalizing (A-24) to K regions and writing each row in scalar form we get
\[
ds_{i} = (-\delta_{i} S_{i} + \sum_{j \neq i} \gamma_{ij} S_{j}(t) + \beta_{i} \sqrt{R_{i}(t)} + \alpha_{i} \sqrt{N(t)})dt + \sum_{j \neq i} \tilde{\sigma}_{ij} d\tilde{W}_{j}(t),
\]
where \(i, j = 1, 2, \ldots, K\), and \(\tilde{\sigma}_{ij}\) is the \(ij^{th}\) element of \(B^{-1}\Omega^{-1}\). Thus, letting \(f_{i} = -\delta_{i} S_{i} + \beta_{i} \sqrt{R_{i}} + \alpha_{i} \sqrt{N} + \sum_{j \neq i} \gamma_{ij} S_{j}\), we obtain

Equation (4).
CONTINUOUS VS. DISCRETE TIME MODELS

We transformed Equation (1) to continuous-time for a reason. In discrete-time models, the optimal decisions are solutions to stochastic difference equations; whereas in continuous-time models, we exploit the continuity of time to obtain non-stochastic (i.e., deterministic) differential equations. Due to the deterministic nature, the latter is more likely to yield analytical solutions than the former.

To illustrate this point, consider the scalar version of our dynamic sales model in discrete-time $S_t = \lambda S_{t-1} + \beta \sqrt{u} + \sum_j \gamma_j S_{\mu-1} + \varepsilon_t$, where $\varepsilon_t \sim N(0, \sigma^2)$, and its continuous-time analog $dS = (\beta \sqrt{u} - \delta S + \sum_j \gamma_j S) dt + \sigma dW$, and $\delta = 1 - \lambda$. Managers seek to maximize the total expected profit $J(u_t) = E\left[ \sum_{t=0}^{T} \pi(S_t, u_t) \right]$ or $J(u(t)) = E\left[ \int_0^{T} e^{-\rho t} \pi(S(t), u(t))dt \right]$. To obtain optimal decisions in discrete-time by maximizing $J$, we have to solve the Bellman equation:

$$V(S_t) = \max_{u_t} \left[ \pi(S_t, u_t) + \frac{1}{1+\rho} E(V(S_{t+1})) \right], \quad (A-25)$$

where $V$ is the value function that generates the stochastic difference equation in $S_t$. Because (A-25) depends on the future expectation $E(\cdot)$, we cannot “get rid of the conditional expectation operator” (Chang 2004, p. 118).

In contrast, in continuous-time model, this conditional expectation disappears due to the application of Taylor’s expansion and Ito’s calculus. To see this, we express (A-25) in continuous-time by applying the principle of optimality,

$$V(S_t) = \max_{u_t} \left[ \pi(S_t, u_t) + \frac{1}{1+\rho} \left( E(V(S_{t+1})) \right) \right]. \quad (A-26)$$

Unlike discrete-time (A-25), we can apply Ito’s calculus and explicitly evaluate the conditional expectation $E[V(S + ds, t + dt)]$ to obtain,

$$E[V(S + ds, t + dt)] = E[V(S, t) + V_d dt + V^2_d ds + 0.5V_{dd} (ds)^2 + 0.5V_{dr} (ds)(dt) + 0.5V_{rr} (dt)^2 + o(dt)]$$

$$= V(S, t) + V_d dt + V^2_d ds + 0.5V_{dd} E[(ds)^2] + 0.5V_{dr} E[(ds)(dt)] + 0.5V_{rr} E[(dt)^2] + o(dt)$$

$$= V(S, t) + V_d dt + V^2_d ds + 0.5V_{dd} \sigma^2 dt + o(dt), \quad (A-27)$$

where the first equality follows from the Taylor’s expansion with $V_x = \partial V / \partial x, V_{xx} = \partial^2 V / \partial x^2$, and $V_{xy} = \partial^2 V / \partial x \partial y$, and the last equality holds because $E[ds] = (\beta \sqrt{u} - \delta S) dt$, $E[(dt)^2] = 0$, $E[(ds)^2] = \sigma^2 dt$, and $E[dsdt] = 0$ from stochastic calculus. Substituting (A-27) into (A-26) and taking limits as $dt \to 0$, we obtain the non-stochastic partial differential equation,

$$-V_t = \max_{u_t} \left[ e^{-\rho t} \pi(S, u_t) + V_S (\beta \sqrt{u} - \delta S + \sum_j \gamma_j S_j) + 0.5V_{ss} \sigma^2 \right]. \quad (A-27)$$

Its corresponding multi-state, multi-control version with spatial and serial correlations is Equation (6), which can be solved analytically (see the proof of Proposition 1 in the Web Appendix).

In sum, continuous-time formulation gets rid of the conditional expectation in (A-26), converts the stochastic problem into a deterministic one, and thus yields optimal decisions that hold across the parameter space (i.e., all feasible values rather than the specific estimated values).
REFERENCES

