Real interest rates, leverage, and bank risk-taking

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Abstract

Do low interest rate environments lead to greater bank risk-taking? We show that, when banks can adjust their capital structures, reductions in real interest rates lead to greater leverage and higher risk for any downward sloping loan demand function. However, if the capital structure is fixed, the effect depends on the degree of leverage: following a decrease in interest rates, well capitalized banks increase risk, while highly levered banks may decrease it if loan demand is linear or concave. Further, the capitalization cutoff depends on the degree of bank competition. This effect therefore should vary across countries and over time.

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1. Introduction

The recent global financial crisis has brought the relationship between interest rates and bank risk taking to the forefront of the economic policy debate. Many observers have blamed the low interest rate environment in the middle part of the recent decade for the credit boom and the ensuing crisis in the late 2000s. In the run up to the crisis, they argue, low interest rates and abundant liquidity led financial intermediaries to take excessive risks by fueling asset prices and promoting leverage. One contributing factor that has been much discussed relates to the role of loose monetary policy: the argument is that had monetary authorities raised interest rates earlier and more aggressively, the consequences of the bust would have been much less severe. More recently, a related debate has been raging on whether continued exceptionally low interest rates are setting the stage for the next financial crisis.1

These arguments have become increasingly popular in both academia and the business press. Surprisingly, however, the theoretical foundations for these claims have not been much studied and hence are not well understood. Macroeconomic models have typically focused on the quantity rather than the quality of credit (e.g., the literature on the bank lending channel) and have mostly abstracted from the notion of risk. Papers that consider risk (e.g., financial accelerator models in the spirit of Bernanke and Gertler[12]) explore primarily how changes in interest rates affects the riskiness of borrowers rather than the risk attitude of the banking system.2 In contrast, excessive risk-taking by financial intermediaries operating under limited liability and asymmetric information has been the focus of a large banking literature which, however, has largely ignored the role of real interest rates and its determinants, such as monetary policy.3 This paper is an attempt to fill this gap.

We develop a model of financial intermediation where banks can engage in costly monitoring to reduce the credit risk in their loan portfolios. Monitoring effort and the pricing (i.e., interest rates) of bank assets and liabilities – debt and equity – are endogenously determined and, in equilibrium, depend on a reference (or risk-free) real interest rate. We start by studying the case where banks’ capital structure is endogenously determined by allowing banks to adjust their capital structure in response to changes in risk-free rates. We obtain two main findings. First, a reduction in risk-free interest rates leads banks to increase their leverage. Reflecting this increase in leverage, our second main finding is that a drop in risk-free rates will unambiguously lower bank monitoring and increase risk taking. We then consider the case where a bank’s capital structure is fixed exogenously and find that, in contrast to the case where capital is optimally chosen, the effects of changes in the risk-free interest rate on bank monitoring and, hence, portfolio risk critically depend on a bank’s leverage. Specifically, a reduction in the risk-free rate leads highly capitalized banks to monitor less, while the opposite is true for poorly capitalized banks when loan demand is linear.

Our model is based on two standard assumptions. First, banks are protected by limited liability and choose the degree to which to monitor their borrowers or, equivalently, choose the riskiness of their portfolios. Since monitoring effort is not observable, a bank’s capital structure can affect its risk-taking behavior. Second, the cost of a bank’s liabilities is affected by changes in the risk

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1 See, for example, Rajan [43], Taylor [47], or Borio and Zhu [17].
2 Angeloni and Faia [10] is a recent attempt to introduce bank risk in a New Keynesian macro framework.
3 Diamond and Rajan [27] and Farhi and Tirole [29] are recent exceptions, although these deal with the effects of expectations of a “macro” bailout rather than the implications of the monetary stance. Reviews of the older literature are in Boot and Greenbaum [16], Bhattacharya, Boot, and Thakor [14], Carletti [21].
free rate. Under these two assumptions, we show that the balance of three coexisting forces – interest-rate pass-through, risk shifting, and leverage – determines how changes in interest rate conditions affect a bank’s risk taking.

The first important determinant of banks’ risk taking decisions is a pass-through effect that acts through the asset side of a bank’s balance sheet. In our model, a reduction in the reference real interest rate is reflected in a reduction of the interest rate on bank loans. This, in turn, lowers the bank’s gross return conditional on its portfolio repaying, reducing the incentive for the bank to monitor. This effect is akin to the portfolio reallocation effect present in portfolio choice models. In these models, when the real yield on safe assets declines, banks will typically increase their demand for risky assets.4

The second effect is a standard risk-shifting problem that operates through the liability side of a bank’s balance sheet. Everything else equal, a drop in the reference interest rate which reduces the cost of a bank’s liabilities will increase the bank’s profit when it succeeds and thus creates an incentive to limit risk taking in order to reap those gains. The extent of this effect, however, depends on the degree of limited liability protection afforded to the bank.5 To see why, consider a fully leveraged bank that is financed entirely through deposits/debt. Under limited liability, this bank will suffer no losses in case of failure. A drop in the cost of its liabilities will increase the bank’s expected net return on all assets by lowering the rate it has to pay on deposits. The bank can maximize this effect by reducing the risk of its portfolio, choosing a safer portfolio for which there is a higher probability the bank will have to repay depositors. In contrast, for a bank fully funded by capital, the effect of a decrease in the cost of its funding will, all other things equal, increase the expected net return uniformly across portfolios and have no effect on the bank’s risk choices.

When banks’ capital structures are exogenously determined, the net effect of a change in the risk-free real interest rate on bank monitoring depends on the balance of these two effects. This, in turn, depends on a bank’s capital structure as well as the structure of the market in which it operates. The risk-shifting effect is stronger the more beneficial is the limited liability protection to the bank. This effect is therefore greatest for fully leveraged banks, and is lowest for banks with zero leverage who as a result have no limited liability protection. In contrast, the magnitude of the pass-through effect depends on how rate changes are reflected in changes to lending rates. Thus, the magnitude of this effect depends on the market structure of the banking industry; it is minimal in the case of a monopolist facing an inelastic demand function, when the pass-through onto the lending rate is zero; and it is maximal in the case of perfect competition, when lending rates fully reflect policy rate changes. It follows that the net effect of a change in interest rate conditions may not be uniform across times, banking systems or individual banks. Following a drop in the risk-free interest rate, monitoring will decrease when leverage is low and increase when leverage is high. The position of this threshold level of leverage depends on the market structure of the banking industry.

By contrast, a third force comes into play when banks can optimally adjust their capital structure in response to changes in interest rate conditions. On the one hand, banks have an incentive to be levered since holding capital is costly. On the other hand, capital serves as a commitment device to limit risk taking and helps reduce the cost of debt and deposits. Banks with limited liability

4 The exception would be banks with decreasing absolute risk aversion who, instead, would decrease their holdings of risky assets Fishburn and Porter [30].

5 This is similar to what happens in models that study the effects of competition for deposits on bank stability (Hellmann, Murdock, and Stiglitz [33], Matutes and Vives [39], and Cordella and Levy-Yeyati [22]).
tend to take excessive risk since they do not internalize the losses they impose on depositors and bondholders. Bank capital reduces this agency problem: the more the bank has to lose in case of failure, the more it will monitor its portfolio and invest prudently. When investors cannot observe a bank’s monitoring but can only infer its equilibrium behavior, higher capital (i.e., lower leverage) will lower their expectations of a bank’s risk-taking and, thus, reduce the bank’s cost of deposits and debt. Given that a drop in the reference interest rate reduces the agency problem associated with limited liability, it follows that the benefit from holding capital will also be reduced. In equilibrium, therefore, lower interest rates will be associated with greater leverage. This result provides a simple micro-foundation for the empirical regularities documented in recent papers, such as in Adrian and Shin [5]. The addition of this “optimal leverage” effect tilts the balance of the other two effects: all else equal, more leverage means more risk taking. Our model’s unambiguous prediction when banks’ capital structures are endogenous is consistent with the claim that lower real interest rates, such as those induced by monetary easing, lead to greater risk taking.

Our contribution to the existing literature is twofold. First, we provide a model that isolates the effect of changes in the reference rate on bank risk taking independently of other macroeconomic considerations related to asset values, liquidity provision, etc. The model provides a theoretical foundation for some of the regularities recently documented in the empirical literature, including the inverse relationship between monetary conditions and leverage, and the tendency for banks to load up on risk during extended periods of low interest rates. Our paper can thus help bridge the gap between macroeconomic and banking models. Second, our framework can help reconcile the somewhat dichotomous predictions of two important strands of research: the literature on the flight to quality (see for example, Bernanke, Gertler, and Gilchrist [13], and Bernanke and Gertler [12]) and that on risk shifting linked to limited liability (for example, Matutes and Vives [39], and Repullo [44]). The paper also contributes to the ongoing policy debate on whether macroprudential tools should complement financial and monetary policy to safeguard macrofinancial stability. We discuss this issue and other policy implications further in the concluding section.

The paper proceeds as follows: Section 2 presents a brief survey of related theoretical work. Section 3 introduces the model and examines the equilibrium. Section 4 solves the case where capital structure is fixed and exogenous. Section 5 examines the role of deposit insurance, different market structures, and other potential feedback effects. Section 6 presents some numerical examples. Section 7 discusses recent empirical findings consistent with the results of the model. Section 8 concludes. Proofs are mostly relegated to Appendix A.

2. Related literature

Our paper is related to a well established literature studying the effects of changes in reference real interest rates on credit markets. The literature on financial accelerators posits that monetary policy tightening, by increasing risk-free interest rates, leads to more severe agency problems by depressing borrowers’ net worth (see, e.g., Bernanke and Gertler [12], and Bernanke, Gertler, and Gilchrist [13]). The result is a flight to quality: firms more affected by agency problems will find it harder to obtain external financing. However, this says little about the riskiness of the marginal borrower that obtains financing because monetary tightening increases agency problems across the board, not just for firms that are intrinsically more affected by agency problems. Thakor [48] focuses on the quantity rather than the quality of credit. Yet, his model has implications for bank risk taking. In [48], banks can invest in government securities or extend loans to risky
entrepreneurs. The impact of monetary policy on the quantity of bank credit and thus on the riskiness of banks’ portfolios depends on its relative effect on banks’ intermediation margins on loans and securities. While the impact on portfolio risk is not explicitly studied, if the rate on securities falls more than that on deposits, the opportunity cost of extending loans would fall and the portion of a bank’s portfolio invested in loans would increase; otherwise, the opposite would happen.

Rajan [42] identifies, in the “search for yield,” a related mechanism through which changes in the reference rate may affect risk taking. He argues that financial institutions may be induced to switch to riskier assets when a monetary easing lowers the yield on their short-term assets relative to that on their long-term liabilities. If yields on safe assets remain low for a prolonged period, continued investment in safe assets will mean that a financial institution will need to default on its long-term commitments. A switch to riskier assets (and higher yields) may increase the probability that it will be able to match its obligations. Dell’Ariccia and Marquez [24] find that when banks face an adverse selection problem in selecting borrowers, policies that lead to reductions in banks’ costs of funds, such as international capital inflows or monetary expansions, may also lead to a credit boom and lower lending standards. This is because banks’ incentives to screen out bad borrowers are reduced when their costs of funds are lowered.

More recently, Farhi and Tirole [29] and Diamond and Rajan [27] have examined the role of “macro bailouts” and collective moral hazard on banks’ liquidity decisions. When banks expect a strong policy response by the monetary authorities should a large negative shock occur (a mechanism often referred to as the “Greenspan put”), they will tend to take on excessive liquidity risk. This behavior, in turn, will increase the likelihood that the central bank will indeed respond to a shock by providing the necessary liquidity to the banking system. Unlike in this paper, their focus is on the reaction function of the central bank (the policy regime) rather than on the level of real interest rates, which may reflect the policy stance. Agur and Demertzis [7] present a reduced form model of bank risk taking to focus on how monetary policymakers should balance the objectives of price stability and financial stability. Drees et al. [28] find that the relationship between the interest rates and risk taking depends on whether the primary source of risk is the opaqueness of a security or the idiosyncratic risk of the underlying investment.

Our paper also relates to a large theoretical literature examining the effects of limited liability, leverage, and deposit rates on bank risk taking. Several papers (e.g., [39,33,22,44,18]) have focused on how competition for deposits (i.e., higher deposit rates) exacerbates the agency problem associated with limited liability and may inefficiently increase bank risk taking. This effect is similar to the risk-shifting effect identified in this paper: more competition for deposits increases the equilibrium deposit rate, compressing intermediation margins and thus reducing a bank’s incentives to invest in safe assets.

The framework we use is based on Dell’Ariccia and Marquez [25] and Allen, Carletti, and Marquez [8]. In particular, the latter shows how banks may choose to hold costly capital to reduce the premium demanded by depositors. They, however, ignore the effects of interest rate conditions and do not examine how leverage moves in response to rate changes. Our result that leverage is decreasing in the reference interest rate is also related to that in Adrian and Shin [4]. In their paper, leverage is limited by the moral hazard induced by the underlying risks in the

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6 Boyd and De Nicolo [18] also show that when moral hazard on the borrowers side is taken into account, the result may be reversed.
environment. In our model, an increase in the risk-free interest rate exacerbates the agency problem associated with limited liability, which in turn leads to a reduction in leverage.

We review the related empirical literature in Section 7.

3. A simple model of bank risk taking

Banks face a negatively sloped demand function for loans, $L(r_L)$, with $L' \leq 0$, where $r_L$ is the gross interest rate the bank charges on loans. In Section 5.3 we examine the impact of alternative market structures.

Loans are risky and a bank’s portfolio needs to be monitored to increase the probability of repayment. The bank is endowed with a monitoring technology, allowing the bank to exert monitoring effort $q$ which also represents the probability of loan repayment. This monitoring effort entails a cost equal to $\frac{1}{2}cq^2$ per dollar lent.

Bank owners/managers raise deposits (or more generally issue debt liabilities) and invest their own money to fund the bank’s loan portfolio. Let $k$ represent the portion of bank assets financed with the bank owner’s money (consistent with other models, this can be interpreted as the bank’s equity or capital), and $1-k$ the fraction of the bank’s portfolio financed by deposits. As we describe below, banks set $k$ optimally to maximize their profits. Note that existing bank regulations support treating banks’ capital choice as endogenous. First, banks with reported capital ratios close to the required regulatory minimum tend to overstate capital by not recognizing losses, especially during financial crises (Peek and Rosengren [41], Huizinga and Laeven [34]). As a result, their true leverage may be higher than the regulatory limit. Second, international capital rules (as set forth in the various Basel accords) allow for regulatory arbitrage of capital requirements, allowing banks to save on capital while taking on more risks, effectively increasing unadjusted leverage. Either way, these bank behaviors call for an endogenous treatment of leverage.

Banks are protected by limited liability and repay depositors only in case of success. Let $r^*$ be the economy’s reference rate in real terms, which for simplicity and without loss of generality can be normalized to be the real risk-free interest rate (we will use “reference” and “risk free” interchangeably). A fraction $1-\alpha$ of deposits are insensitive to bank risk-taking and receive a deposit rate equal to the reference rate, so that $r_D = r^*$. This is consistent with the existence of deposit insurance for a fraction $1-\alpha$ of deposits, so that depositors are not concerned about being repaid by the bank, but nevertheless want to receive a return that compensates them for their opportunity cost, which would be incorporated in the interest rate $r^*$. The remaining fraction, $\alpha$, are sensitive to risk (e.g., are uninsured) and must be compensated for the bank’s expected risk taking. While depositors cannot directly observe $q$ (i.e., $q$ is chosen after deposits have been raised), from observing the capital ratio $k$ they can nevertheless infer the bank’s equilibrium monitoring behavior, $\hat{q}$. Given an opportunity cost of $r^*$, risk sensitive depositors will demand a promised repayment $r_D$ such that $r_D E[q|k] = r^*$, or in other words $r_D = r^* E[q|k]$. From the bank’s perspective, this means that, given a quantity of loans $L$ and a ratio of capital $k$, a total of $kL$ of deposits will be raised. The total cost of these deposits will then be $\alpha (1-k) L r^* E[q|k] + (1-\alpha)(1-k) L r^* = (\alpha E[q|k] + (1-\alpha)) r^* (1-k) L$.

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7 The assumption of a downward sloping demand curve for loans is supported by broad empirical evidence (e.g., Den Haan, Summer, and Yamashiro [26]).

8 For a model in the same spirit but where banks choose among portfolios with different risk/return characteristics, see Cordella and Levy-Yeyati [22].

9 Keeley [37] formally shows that when deposits are fully protected by deposit insurance, the supply of deposits will not depend on bank risk.
Equity, however, is more costly, with a yield \( r_E = \frac{r^* + \xi}{q} \), with \( \xi \geq 0 \). The cost \( r_E \) can be interpreted as the opportunity cost for the bank owner/manager of investing in the bank, adjusted to reflect the bank’s risk through the probability of success \( q \).\(^{10}\) The term \( \xi \) represents an equity premium in line with existing literature (see for instance, [33,44,25,8]).

We structure the model in three stages. For a fixed reference interest rate \( r^* \), in stage 0 banks choose their desired capitalization ratio \( k \). Unsecured investors observe the bank’s choice of \( k \) and set the interest rate they charge on the bank’s liabilities. In stage 1, banks choose the interest rate to charge on loans, \( r_L \). In stage 2, banks then choose how much to monitor their portfolio, \( q \).

3.1. Equilibrium

We focus first on the case where all deposits are risk-sensitive (\( \alpha = 1 \)). We discuss the role of deposit insurance (\( \alpha < 1 \)) in Section 5.1. We solve the model by backward induction, starting from the last stage. The bank’s expected profit can be written as:

\[
\Pi = \left( q \left( r_L - r_D(1-k) - r_E k \right) - \frac{1}{2} cq^2 \right) L(r_L),
\]

which reflects the fact that the bank’s portfolio repays with probability \( q \). When the bank’s projects succeed, the owner (e.g., shareholders) receives a per-loan payment of \( r_L \) and earns a return \( r_L - r_D(1-k) \) after repaying depositors. When the bank fails, the owner receives no revenue but, because of limited liability, does not repay depositors. The term \( r_E k \) represents the opportunity cost of the bank’s owner/manager, adjusted for the bank’s probability of success \( q \).\(^{11}\)

We can rewrite (1) as

\[
\Pi = \left( q \left( r_L - r_D(1-k) \right) - (r^* + \xi)k - \frac{1}{2} cq^2 \right) L(r_L).
\]

Maximizing (2) with respect to \( q \) yields

\[
\hat{q} = \min \left\{ \frac{r_L - r_D(1-k)}{c}, 1 \right\}.
\]

In the absence of deposit insurance, depositors will demand a promised return of \( r_D = r^* E[q|k] \), as argued above. Since depositors’ expectations about bank monitoring, \( E[q|k] = \hat{q}(r_D|k) \), must in equilibrium be correct, we can substitute \( E[q|k] = \hat{q}(r_D|k) \) into the equation above to solve for the equilibrium level of monitoring \( \hat{q} \):

\[
\hat{q}(k) = \frac{1}{2c} \left( r_L + \sqrt{r_L^2 - 4cr^*(1-k)} \right).
\]

\(^{10}\) We assume that the premium on equity, \( \xi \), is independent of the real interest rate \( r^* \). However, from an asset pricing perspective these are likely to be correlated through underlying common factors which may drive the risk premium as well as the risk free rate. Our results continue to hold as long as the within period correlation between \( \xi \) and \( r^* \) is sufficiently different from (positive) one.

\(^{11}\) Equivalently, one can interpret \( r_E \) as the required return on equity (corrected for risk). In this case, expected profits must be greater than or equal to \( (r^* + \xi)kL \) in order for equity investors to be willing to provide financing, or:

\[
\Pi = \left( q \left( r_L - r_D(1-k) \right) - \frac{1}{2} cq^2 \right) L(r_L) \geq (r^* + \xi)kL(r_L).
\]

Subtracting \( (r^* + \xi)kL(r_L) \) from both sides yields the exact expression in the text.
From (4) it is immediate that, holding everything else constant, an increase in the risk free rate would lead to a reduction in the level of monitoring. This is the classic risk-shifting effect identified by the literature on banking competition (see, for instance, [33,39,44]). Here, however, bank lending rates and leverage are also endogenous and their movements in reaction to a change in the policy rate will affect monitoring.

We can now solve stage 1 where banks choose the loan interest rate. Assuming that an interior solution exists, we substitute \( \hat{q} \) into the expected profit function and obtain\(^{12}\):

\[
\Pi(\hat{q}) = \left( \frac{(r_L - r_D(1-k))^2}{2c} - (r^* + \xi)k \right) L(r_L).
\]

Maximizing (5) with respect to the loan rate yields the following first order condition:

\[
\frac{\partial \Pi(\hat{q})}{\partial r_L} = L(r_L) \frac{r_L - r_D(1-k)}{c} + \frac{\partial L(r_L)}{\partial r_L} \left( \frac{r_L - r_D(1-k)}{2c} \right)^2 - (r^* + \xi)k \frac{\partial L(r_L)}{\partial r_L} = 0.
\]

The equilibrium lending rate, \( \hat{r}_L \), is increasing in the risk-free rate \( r^* \) (see the proof of Lemma 1). This provides a second channel through which changes in the risk-free rate affect the equilibrium monitoring effort. This “pass-through” effect acts opposite to the classic risk-shifting one. An increase in \( r^* \) is reflected in a higher lending rate. This, in turn, increases the bank’s gross return conditional on its portfolio repaying, providing greater incentives for the bank to monitor. This effect is akin to the portfolio reallocation effect present in portfolio choice models.

We can now maximize bank profits with respect to the capital ratio \( k \) in stage 0:

\[
\max_k \Pi = \left( \hat{q}(\hat{r}_L - r_D(1-k)) - (r^* + \xi)k - \frac{1}{2}c\hat{q}^2 \right) L(\hat{r}_L),
\]

subject to the equilibrium condition that the deposit rate compensate unsecured investors for the expected risk-taking by the bank, \( r_D = \frac{r^*}{E[q|k]} \), where \( \hat{q} = \hat{q}(r_L; k) \) is the equilibrium choice of monitoring induced by the bank’s choice of the loan rate \( r_L \) and capitalization ratio \( k \), and \( \hat{r}_L = \hat{r}_L(k) \) is the optimal loan rate given \( k \). In other words, the bank takes into account the influence of its choice of \( k \) on its subsequent loan pricing and monitoring decisions. In equilibrium, it must also be true that \( E[q|k] = \hat{q} \), so that investors’ expectations are correct. The first order condition for \( k \) can be expressed as

\[
\frac{d\Pi}{dk} = \frac{\partial \Pi}{\partial k} + \frac{\partial \Pi}{\partial r_L} \frac{dr_L}{dk} + \frac{\partial \Pi}{\partial q} \frac{dq}{dk} = \frac{\partial \Pi}{\partial k} = 0
\]

since the last two terms are zero from the envelope theorem. Substituting, this becomes

\[
\frac{d\Pi}{dk} = \left( (r_L - cq) \frac{\partial q}{\partial k} - \xi \right) L(r_L) = 0,
\]

which characterizes the bank’s optimal choice of \( \hat{k} \). As we show in the next proposition, \( \hat{k} \) is strictly positive for a broad range of parameter values.

**Proposition 1.** Equilibrium bank leverage decreases with the real interest rate: \( \frac{d\hat{k}}{dr^*} > 0 \).

\(^{12}\) It is straightforward to see that there always exist values of \( c \) that guarantee an interior solution for \( q \). Later, we demonstrate numerically that an interior solution to the model exists. In other words, there is a wide range of parameter values for which the first order conditions characterize the equilibrium.
The proposition establishes that, when an internal solution \( \hat{k} \) for the capitalization ratio exists, then \( \hat{k} \) will be increasing in \( r^* \). Put differently, a drop in the reference interest rate will induce banks to be more leveraged (i.e., to hold less capital).

An increase in the reference interest rate, which reflects investors’ opportunity costs, increases the rate the bank has to pay on its debt liabilities and exacerbates the bank’s agency problem – note that at \( r^* = 0 \), a limit case where the principal is not repaid at all, there is no moral hazard and \( \hat{q} = q(k = 1) = \frac{rL}{c} \), the level of monitoring for a pure equity financed bank. This effect is essentially the same as in the flight to quality literature (see for example [12]). It follows that as the reference real interest rate increases so does the benefit from holding capital, the only commitment device available to the bank to reduce moral hazard. Put differently, investors will allow banks to be more levered when the reference interest rate is low relative to when it is high.

A similar result is in [4], where leverage is a decreasing function of the moral hazard induced by the underlying risks in the environment. Evidence of this behavior is documented in [5]. Note that since monitoring depends, among other things, on leverage, this result establishes a third channel through which the reference rate affects \( q \).

The following result characterizes banks’ loan pricing decisions as a function of the interest rate, and will be useful in establishing the next main result.

**Lemma 1.** When bank leverage, the loan rate, and the level of monitoring are all optimally chosen with respect to the real interest rate \( r^* \), the optimal loan rate \( \hat{r}_L \) is increasing in \( r^* \):

\[
\frac{d\hat{r}_L}{dr^*} \geq 0,
\]

with the inequality strict whenever the optimal loan rate is not at a boundary.

The intuition for the lemma is straightforward: when the interest rate increases, this raises the opportunity cost on all forms of financing. Consequently, in equilibrium the rate that the bank charges on any loans also increases. In other words, there is at least some pass through of the changes in the bank’s costs of funds onto the price of bank credit extended, which is reflected in a higher loan rate. However, interest margin \( r_L - r_D \) nevertheless gets compressed as a result of an increase in the interest rate \( r^* \).

We can now state our next main result:

**Proposition 2.** When bank leverage is optimally chosen to maximize profits, monitoring will always increase with the real interest rate:

\[
\frac{d\hat{q}}{dr^*} > 0.
\]

This proposition shows that the net effect of an increase in the reference rate (the combination of the risk-shifting effect, the pass-through effect, and the leverage effect) is to increase equilibrium monitoring. Conversely, reductions in \( r^* \), such as those that accompany monetary easing or financial liberalizations, should lead to more highly levered banks and reduced monitoring effort.

A corollary to these results is that the equilibrium volume of credit extended, \( L(\hat{r}_L) \), is decreasing in the interest rate \( r^* \). This follows from Lemma 1. As in most models, a reduction in the reference rate leads to an increase in aggregate lending. Here, however, the expansion in credit is coupled with a deterioration of banks’ balance sheets. Indeed, from Proposition 2 we know that a decrease in the interest rate leads to greater leverage, less monitoring, and thus riskier bank portfolios.

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13 Berlin and Mester [11] show that markups on loans decrease as market rates increase, implying that increases in market rates translate into less than one-for-one increases in loan rates.
It bears emphasizing that the clear cut effect of a change in the interest rate arises only when banks are able to adjust their capital structures (i.e., $k$) in response to changes in $r^*$. In Section 4 we examine the case in which banks are unable to adjust their capital structure. We show that, absent the leverage channel, the net effect of reference rate changes (the balance between the risk-shifting and pass-through effect) depends on the degree of bank capitalization.

3.2. A risk shifting interpretation

Before moving on to study the case where bank capital structure is exogenous, it is worth mentioning that the model of bank monitoring described above can be alternatively cast as a more classic risk-shifting problem. Suppose that there is a conflict of interest between bondholders and shareholders, in that shareholders can choose between investments that have a lower probability of success, but that pay off more conditional on success. Specifically, assume that banks have access to a continuum of portfolios characterized by a parameter $q \in [0, 1]$, with returns $r_L - \frac{1}{2} cq$ and probability of success $q$. As above, banks face a negatively sloped demand function for loans, $L(r_L)$, where $r_L$ is the gross interest rate the bank charges on loans. Banks choose $q$ and $r_L$ and are financed by a fraction $k$ of equity and a fraction $1 - k$ coming from debt (i.e., deposits), also exactly as above. Note that lower $q$ implies a higher return conditional on success, but a lower probability of success.

With this alternative interpretation of the risk choice $q$, the bank’s payoff is again given exactly by (1). Greater capital leads to less risk taking (higher $q$), as in (3). This means that the solution to this problem is identical to that presented in Section 3.1, and that all results continue to hold exactly as stated.

4. Equilibrium when leverage is exogenous

So far, we have assumed that banks are able to adjust their capital structure freely, thus adopting the degree of leverage that is optimal given the reference interest rate used for pricing all assets and liabilities. However, it is possible that some banks may face difficulties adjusting their capital structure, at least in the near term. This could be the case if, for instance, contractionary periods for the economy also increase banks’ costs of adjusting the liability side of their balance sheets, thus making them less willing or able to optimal adjust their degree of leverage. It may similarly be true for individual banks that would optimally like to choose a level of capital below the regulatory minimum. For such banks, changes in the policy rate would not be reflected in their capitalization decisions since the regulatory constraint would be binding. In this section we study the case where banks have a fixed capital structure, $k$, and can only adjust their balance sheets through the interest rate they charge (and, by extension, the size of their loan portfolio) and the degree of monitoring.

Formally, assume that for any specific bank, $k$ is given and exogenous. This is equivalent to eliminating stage 0 in the model above. We solve the model by backward induction, starting from the last stage. The bank’s expected profit is given as above by (1). Taking the loan rate $r_L$ as given, we can solve for the optimal degree of monitoring, which yields the same expression as in (3). From (4) we obtain immediately, as before, that the direct (i.e., for a given lending rate) effect of an increase in the reference interest rate $r^*$ on bank monitoring is non-positive, $\frac{\partial q}{\partial r^*} \leq 0$.

We can now solve stage 1 where banks choose the loan interest rate. Again assuming that an interior solution exists, we can state the following result.
Proposition 3. There exist a degree of capitalization, $0 \leq \tilde{k} < 1$, such that, for $k \geq \tilde{k}$, and assuming an interior solution for the optimal loan rate $r_L$, bank monitoring increases with the real interest rate, $\frac{d\hat{q}}{dr^*} > 0$.

The intuition behind this result is that an increase in the reference interest rate leads to an increase in both the interest rate a bank charges on its loans (i.e., $\frac{d\hat{r}_L}{dr^*} > 0$) and that which it pays on its liabilities, $r_D$. The first effect, which reflects the pass-through of the bank’s cost of funds on loan rates, increases the incentives to monitor. The second effect, the risk-shifting effect, decreases monitoring incentives to the extent that it applies to liabilities that are repaid only in case of success. While an increase in the bank’s cost of funds leads to a compression of the intermediation margins, $r_L - r_D$, the overall effect on a bank’s risk-taking decision depends on how well capitalized the bank is. This is because banks, in contrast to the analysis in Section 3.1, are constrained in their ability to adjust their leverage. From (3) it is evident that for a bank funded entirely through capital, so that $k = 1$, the risk-shifting effect disappears. In this case, an increase in the interest rate $r^*$ increases the level of bank monitoring $\hat{q}$. Therefore, banks with a low leverage ratio will react to an increase in the reference interest rate by taking on less risk.

For $k < 1$, however, an increase in the interest rate on deposits will have a direct negative impact on $\hat{q}$. Thus, it is possible that, for a bank entirely funded with deposits, the risk-shifting effect could dominate due to the compression in the net interest margin $r_L - r_D$. In fact, it is possible to show that for poorly capitalized banks, increases in the real interest rate lead to less monitoring as long as loan demand is not too convex around the optimal interest rate. This can be illustrated for the case where loan demand is linear: $L = A - br_L$.\(^{14}\)

Corollary 1. If loan demand is linear, i.e., $L = A - br_L$, then: (1) the threshold $\tilde{k}$ defined in Proposition 3 is strictly positive: $\tilde{k} > 0$; and (2) for $k < \tilde{k}$, bank monitoring decreases with the real interest rate, $\frac{d\hat{q}}{dr^*} < 0$.

The proof of Proposition 3 also shows that, as in the previous section, a link exists between the interest rate and total bank credit. Since the equilibrium loan rate, $\hat{r}_L$, is increasing in the interest rate, the total volume of credit extended, $L(\hat{r}_L)$, will be decreasing in $r^*$. Therefore, a decrease in $r^*$ leads to an increase in bank credit, as expected. Interestingly, however, such an expansion of credit need not be coupled with riskier bank balance sheets since, from Corollary 1, we know that bank monitoring increases for banks with a relatively low level of capital when loan demand is not too convex.\(^{15}\)

5. Extensions

5.1. Deposit insurance

So far, we have considered only the case where all deposits are uninsured ($\alpha = 1$), so that all depositors must be compensated for the amount of risk they anticipate the bank will take. This

\(^{14}\) Linearity is a sufficient (but not necessary) condition for this result. The proposition holds for other loan demand functions as long as they are not “too convex”.

\(^{15}\) This does not mean, however, that the expansion in credit induced by a drop in $r^*$ implies a safer banking system. While poorly capitalized banks monitor more when $r^*$ falls, they still are riskier than banks with higher levels of capital. The aggregate effect, therefore, is ambiguous.
helps clarify that our results are not driven by a bank’s desire to exploit the deposit insurance system as real interest rates change. In practice, however, many bank deposits are insured, which corresponds to the case where \( \alpha \in [0, 1) \). In this section, we show that all of our qualitative results continue to hold even for the case where all deposits are insured (i.e., for \( \alpha = 0 \)) when bank leverage is exogenous. When banks can adjust their leverage, however, the existence of deposit insurance does play an important role as it reduces a bank’s incentive to commit itself to monitor through a higher level of capital.

Suppose now that a fraction \( 1 - \alpha > 0 \) of deposits are insured and thus insensitive to bank risk taking. As a consequence, these deposits need be promised only \( r^* \), while uninsured deposits must be promised \( r^*_{\text{D}}(1-k) \). Denote the average deposit rate by \( \tilde{r}_D = (\alpha \frac{1}{L(q)} + (1-\alpha))r^* \). As before, we solve the model by backward induction. The solutions for the last two stages are analogous to those in the previous section. Assuming an interior solution, the optimal degree of monitoring is given by \( \tilde{q} = \frac{r_L - r_D(1-k)}{c} \), as usual. Since in equilibrium depositors’ expectations must be correct, we can substitute for \( \tilde{r}_D \) and solve for \( \tilde{q} \) as
\[
\tilde{q} = \frac{1}{2c}(r_L - (1-\alpha)r^*(1-k) + \sqrt{(r_L - (1-\alpha)r^*(1-k))^2 - 4\alpha cr^*(1-k)}).
\]
Define \( \tilde{r}_L \equiv r_L - (1-\alpha)r^*(1-k) \). We can now write the previous expression as
\[
\tilde{q} = \frac{1}{2c}(\tilde{r}_L + \sqrt{\tilde{r}_L^2 - 4\alpha cr^*(1-k)}),
\]
which is of the same form as (4) for the case where \( \alpha = 1 \). Therefore, the bank’s maximization problem can be solved in the same way as before.

At stage 0, the bank maximizes profits with respect to the capital ratio \( k \):
\[
\max_k \Pi = \left( \tilde{q}(\tilde{r}_L - \tilde{r}_D(1-k)) - (r^* + \xi)k - \frac{1}{2}c\tilde{q}^2 \right) L(\tilde{r}_L).
\]
To see how the existence of deposit insurance affects a bank’s optimal use of leverage, consider the first order condition with respect to \( k \):
\[
\frac{\partial \Pi}{\partial k} = (\alpha - 1)r^* - \xi + \frac{\partial \tilde{q}}{\partial k}(\tilde{r}_L - c\tilde{q}) L(\tilde{r}_L) = 0.
\]
Since \( L(\tilde{r}_L) = 0 \) is not optimal, this is equivalent to the simpler condition
\[
(\alpha - 1)r^* - \xi + \frac{\partial \tilde{q}}{\partial k}(\tilde{r}_L - c\tilde{q}) = 0.
\]
Substituting for \( \tilde{r}_L \), the left hand side of (9) gives
\[
(\alpha - 1)r^* - \xi + \frac{\partial \tilde{q}}{\partial k}(r_L - (1-\alpha)r^*(1-k) - c\tilde{q}).
\]
This expression is increasing in \( \alpha \). This means that, ceteris paribus, there is a greater incentive to hold capital \( k \) the higher is \( \alpha \). Now let \( \alpha \to 0 \). In this case, (10) converges to \( \frac{\partial \Pi}{\partial k} = -r^* - \xi < 0 \). As \( \alpha \to 0 \), the first order condition could never be satisfied, and the solution would have \( k = 0 \). In other words, when all deposits are insured and leverage is endogenous, banks choose maximum leverage. Moreover, since bank profits are continuous in \( k \) and in the fraction of uninsured deposits \( \alpha \), it is straightforward to see that even for some strictly positive fraction of uninsured deposits, \( \alpha > 0, k = 0 \) will still be optimal. (We study how changes in the ratio of insured to uninsured deposits affects bank leverage in Section 6.)
However, when bank leverage is exogenous, so that $k$ is given, letting $\alpha \to 0$ does not change the bank’s problem in any qualitative way, as can be seen from comparing the bank’s profit expression in (8) to that in (1), as well as the solution to the bank’s monitoring decision in (7) compared to that in (4). Therefore, as $\alpha \to 0$, similar results as those obtained in Proposition 3 and Corollary 1 continue to hold.\footnote{A formal derivation of these results follows as a corollary to Proposition 3, and is available upon request. (See previous working paper.)}

5.2. Constraints in raising capital

In this section we briefly study the case where banks are constrained in raising or paying equity and must instead adjust leverage by changing only the asset size of their balance sheets (issues related to balance sheet adjustments have been raised recently by [6], among others). To do so, we modify the setup slightly and assume a bank has a fixed amount of equity $K$ which it cannot change. The capital ratio will then be $k = \min\{\frac{K}{L}, 1\}$. Bank profits can be written as

$$\Pi = \left( q(r_L - r_D(1 - k)) - (r^* + \xi)k - \frac{1}{2}cq^2 \right) L(r_L)$$

$$= \left( q(r_L - r_D(1 - k)) - \frac{1}{2}cq^2 \right) L(r_L) - (r^* + \xi)K.$$  

Since the bank cannot adjust $K$, the bank’s objective reduces to maximizing $\Pi$ with respect to $r_L$ and $q$. As usual, we have that $\hat{q} = \min\{\frac{L(r_L)}{L(r_D)}, 1\}$. Assuming an interior solution for $q$, the first order condition for $r_L$ yields

$$\frac{\partial \Pi(\hat{q})}{\partial r_L} = L(r_L) r_L - r_D(1 - k) \left(1 + r_D \frac{\partial k}{\partial L} \frac{\partial L}{\partial r_L}\right) + \frac{\partial L(r_L)}{\partial r_L} \frac{(r_L - r_D(1 - k))^2}{2c} = 0.$$  

As usual, we can use the Implicit Function Theorem: let $M = \frac{\partial \Pi(\hat{q})}{\partial r_L}$, then $\frac{dr_L}{dr^*} = -\frac{\partial M}{\partial r^*}$. Assuming the second order condition is satisfied, the numerator for the Implicit Function Theorem will be

$$\frac{\partial M}{\partial r^*} = \frac{1}{q} \left[ L(r_L) \left(1 + r_D \frac{\partial k}{\partial L} \frac{\partial L}{\partial r_L}\right) + \frac{r_L - r_D(1 - k)}{c} \frac{\partial L}{\partial L} \frac{\partial L}{\partial r_L}\right]$$

$$- \frac{\partial L(r_L)}{\partial r_L} \frac{r_L - r_D(1 - k)}{c} (1 - k).$$  

(11)

It is straightforward to see that, for any given demand function $L$, as $K$ grows large, $k \to 1$ and (11) becomes

$$\frac{1}{q} \left[ L(r_L) \frac{r_L}{c} \frac{\partial k}{\partial L} \frac{\partial L}{\partial r_L}\right] > 0.$$  

Therefore, $\frac{dr_L}{dr^*} > 0$ for large $K$. But as $k \to 1$, $\text{sign}(\frac{dq}{dr^*}) = \text{sign}(\frac{dr_L}{dr^*})$, implying that $\frac{dq}{dr^*} > 0$ as $k \to 1$.

More generally, consider the case for smaller values of $K$, and let $K \to 0$. In this case, the capital ratio $k$ also converges to 0, implying that $\frac{dq}{dr^*}|_{K=0} = \frac{dq}{dr^*}|_{k=0}$. In other words, for $K$ vanishingly small the effect of an increase in the reference rate $r^*$ is not only qualitatively but
also quantitatively the same as when the capital structure is fixed (i.e., exogenous) and leverage is very high. Therefore, the analysis in this section implies that even though banks may be able to adjust their leverage through (asset side) balance sheet changes, the effect of a constraint on a bank’s ability to raise or pay out equity acts much in the same fashion as the leverage constraint analyzed in Section 4.

5.3. The role of market structure

Here we examine the effect of alternative loan market structures. We look at two diametrically opposed cases: First, a perfectly competitive credit market where banks take the lending rate as given, which is determined by market clearing and a zero profit condition for the banks; and second, a monopolist facing a loan demand function that is perfectly inelastic up to some fixed loan rate \( R \). This upper limit can be interpreted as either the maximum return on projects, or as the highest rate consistent with borrowers satisfying their reservation utilities. For these two alternative structures, we show that our qualitative results when leverage is endogenous continue to hold.

5.3.1. The perfect competition case

Consider the following modification of our model to incorporate perfect competition. At stage 0, given the reference interest rate, the lending rate is set competitively so that banks make zero expected profits in equilibrium. At stage 1, banks choose their desired leverage (or capitalization) ratio \( k \); unsecured investors observe the bank’s choice of \( k \) and set the interest rate they charge on the bank’s liabilities, \( r_D \). And in the last stage, as before, banks choose the extent of monitoring.

Again, we solve the model by backward induction. As for the cases analyzed in Sections 4 and 3.1, solving for the equilibrium monitoring and imposing \( r_D = r^*_E[\hat{q}|k] \) implies, as before

\[
\hat{q} = \frac{r_L + \sqrt{r_L^2 - 4cr^*(1-k)}}{2c}.
\]

We first consider the case where \( k \) is exogenous. For this case, we impose a zero profit condition,

\[
\hat{\Pi} = L \left( \hat{q}r_L - r^* - k\xi - \frac{c\hat{q}^2}{2} \right) = 0,
\]

to obtain \( r_L \) as a function of \( r^* \) and \( k \). Note that in order for an \( r_L \) that satisfies this condition to exist, \( k \) needs to be “large enough”. Let \( k \) indicate this threshold.\(^{17}\)

We can now state the following result.

**Proposition 4.** In a perfectly competitive market, for a fixed capitalization ratio \( k \), bank monitoring increases with the real interest rate, \( \frac{d\hat{q}}{dr^*} > 0 \), for \( k \in [\hat{k}, 1] \).

This result contrasts with that obtained in Proposition 3 for the case where banks have market power. There, the effect of a change in the reference real interest rate on risk taking depended

\(^{17}\) For very low levels of capital, the model entails credit rationing (as in Stiglitz and Weiss [46]) if the lending rate is too low. Put differently, because of the effect of \( r_D \) on \( q \), there does not exist an interest rate at which investors are willing to lend to a bank. From \( \hat{q} = \frac{r_L + \sqrt{r_L^2 - 4cr^*(1-k)}}{2c} \), it is immediate that \( r_L^2 - 4cr^*(1-k) \) needs to be positive, which imposes a lower bound for \( r_L \). Then, one can derive the lower bound for \( k \) for a zero profit equilibrium to exist, which we denote by \( \hat{k} \) (details are in Appendix A).
on the degree of bank capitalization, \( k \). For a sufficiently low level of \( k \), monitoring decreased as the reference interest rate increased. Here, the bank’s response to changes in the reference interest rate in terms of monitoring, \( \frac{dq}{dr^*} \), is always positive, and is increasing in \( k \). This result stems from the fact that the pass-through of the policy rate onto the loan rate is maximum in the case of perfect competition and must perfectly reflect the increase in the policy rate. It follows that the pass-through effect dominates the risk-shifting effect, so that the region where \( \frac{dq}{dr^*} < 0 \) disappears (even for linear demand).

We next endogenize the capital ratio \( k \), as in Section 3.1. Banks maximize

\[
\max_k \Pi = L\left( \hat{q}r_L - r^* - k\xi - \frac{c}{2}\hat{q}^2 \right),
\]

which gives

\[
\hat{k} = 1 - r_L^2 \frac{\xi(r^* + \xi)}{cr(r^* + 2\xi)^2}.
\]

We obtain the lending rate from the zero profit condition for banks:

\[
\widehat{\Pi} = L\left( \hat{q}(\hat{k})r_L - r^* - \hat{k}\xi - \frac{c}{2}\hat{q}^2(\hat{k}) \right) = 0.
\]

From (12) we can solve for the equilibrium lending rate, capital, and monitoring as:

\[
\hat{r}_L = \frac{r^* \xi + (r^*)^2}{2cr^* + (r^*)^2 + 2\xi^2}, \quad \hat{k} = \frac{r^* \xi + (r^*)^2}{3r^* \xi + (r^*)^2 + 2\xi^2}, \quad \hat{q} = \frac{r^*4(r^* + \xi)^2}{2c(3r^* \xi + (r^*)^2 + 2\xi^2)}.
\]

We obtain the following result.

**Proposition 5.** In a perfectly competitive market, equilibrium bank leverage decreases with the real interest rate: \( \frac{dk}{dr^*} > 0 \). And, when bank leverage is optimally chosen to maximize profits, monitoring will always increase with the real interest rate: \( \frac{dq}{dr^*} > 0 \).

This result extends Propositions 1 and 2 to the case of perfect competition and establishes that even when credit markets are perfectly competitive, reductions in reference interest rates in equilibrium lead banks to both hold less capital and take on more risk once one incorporates banks’ ability to adjust their optimal leverage ratios.

5.3.2. A monopolist facing inelastic demand

Here, we assume that there is a fixed demand for loans, \( L \), as long as the lending rate does not exceed a fixed value of \( R \). This setting can be interpreted as one where each borrower has a unit demand for loans and \( R \) is the borrower’s reservation loan rate. Demand becomes zero for \( r_L > R \). This eliminates any pricing effects on loan quantity and allows us to focus on a case where the loan rate is not responsive to changes in the cost of funding. Given the fixed, inelastic demand, it will always be optimal to set \( \hat{r}_L = R \).

We can solve for \( \hat{q} \), imposing the condition that \( r_D = \frac{r^*}{E[q|k]} \), and obtain

\[
\hat{q} = \frac{R + \sqrt{R^2 - 4cr^*(1-k)}}{2c},
\]

from which we can state the following claim.
Claim 1. For \( k \in [0, 1) \) fixed, a monopolist bank facing a demand function that is perfectly inelastic for \( r_L \leq R \) will always decrease monitoring when the real interest rate increases: \( \frac{d\hat{q}}{dr^*}\bigg|_k < 0 \).

For \( k = 1, \frac{d\hat{q}}{dr^*}\bigg|_k = 0 \).

Proof. From (13) we immediately obtain
\[
\frac{d\hat{q}}{dr^*} = -\frac{1-k}{\sqrt{R^2 - 4cr^* (1-k)}} < 0.
\]

This result stands in contrast to that in Proposition 4 for the case of perfect competition when leverage is exogenous. There, irrespective of the level of leverage, risk taking was always decreasing in the interest rate. Here, risk taking is always increasing in the interest rate. The difference stems precisely from the extent to which the bank passes onto the loan rate changes in its costs. If demand is inelastic, the pass-through is zero as the lending rate is always held at its maximum, \( R \), and thus cannot adjust when the interest rate changes. Therefore, the impact of a change in the reference interest rates on monitoring, \( \hat{q} \), operates solely through the liability side of the bank’s balance sheet, reducing the bank’s return in case of success and leading it to monitor less. Put differently, there is only a risk-shifting effect. By contrast, in the perfect competition case the pass-through is at its maximum and the impact of a change in \( r^* \) on the lending rate dominates the risk shifting effect.

To study the effect of a change in the reference interest rate when the monopolist bank can choose the capitalization ratio \( k \), we maximize bank profits with respect to \( k \):
\[
\max_k \Pi = L \left( \hat{q} R - r^* - k \xi - \frac{c}{2} \hat{q}^2 \right).
\]

The solution is
\[
\hat{k} = 1 - R^2 \frac{\xi (r^* + \xi)}{cr^* (r^* + 2\xi)^2}.
\]

We can substitute the solution \( \hat{k} \) back into the formula for \( \hat{q} \) to obtain
\[
\hat{q} = R \frac{(r^* + \xi)}{cr^* (r^* + 2\xi)}.
\]

It is now immediate that Propositions 1 and 2 extend to this case of a pure monopolist: \( \frac{dk}{dr^*} > 0 \) and \( \frac{d\hat{q}}{dr^*} > 0 \) when the bank can adjust its target capital ratio in response to a change in the reference interest rate.

5.4. Potential feedback effects

Ours is a partial equilibrium model. It isolates the effect of changes in the reference rate on bank risk taking independently of other macroeconomic considerations related to asset values, liquidity provision, etc. Yet, by providing a theoretical foundation for some of the regularities recently documented in the empirical literature (an inverse relationship between monetary conditions and leverage, and the tendency for banks to load up on risk during extended periods of low interest rates) it can help bridge the gap between macroeconomic and banking models. That said, a substantial amount of work is needed to incorporate our framework in a general equilibrium model. We see (at least) two directions in which to generalize our framework.

First, one could endogenize the reference rate and consequently the cost of equity. In the model we implicitly assume that both equity and deposits are infinitely elastically supplied at a
fixed rate. One could assume that the economy is partly resource constrained and that the supply of investable funds is upward sloping. Shifters such as monetary policy shocks (open market operations) and exogenous capital flows could then move the supply curve. In such a model, the effects of a shock on \( k \) and \( q \) would be quantitatively smaller than in our partial equilibrium model: for instance, the initial decline in the reference rate (for instance following capital inflows) would be partly undone through the subsequent increase in the demand for loanable funds. But qualitatively our main results would continue to hold.

The analysis would be more complicated if one endogenized the loan demand function. Suppose entrepreneurs could choose between investing in risky projects and a risk free asset. Then, changes in the reference rate, \( r^* \), would simultaneously affect the cost of a bank’s liability and the position of the loan demand function. For instance, an increase in \( r^* \) would make investment in the risk free asset more attractive and decrease the demand for entrepreneurial loans. Everything else equal, this would compress the interest rate pass-through and favor the risk-shifting effect. For a fixed \( k \), the net effect on \( q \) would then depend on how much the demand function shifts with \( r^* \).

6. A numerical example

In this section, we present some simple numerical simulations of the model. The purpose is twofold. First, we want to provide an intuitive graphical illustration of the effects identified in this paper. Second, since most of our analysis relies on internal solutions for several of the choice variables in the model, the example serves to demonstrate that there is a broad set of parameter values for which such solutions indeed exist.

For this example we assume \( L = A - br_L \), with \( A = 100 \) and \( b = 8 \). We also assume that 35 percent of the bank’s liabilities consist of insured deposits and the rest are uninsured and therefore must be priced to reflect their risk. This is to provide some realism to the numbers and also to cover both cases considered in our analysis. Finally, we set the monitoring cost parameter \( c = 9 \) and the equity premium, \( \xi \), to 6 percent.\(^{18}\)

Fig. 1 illustrates Proposition 3. The equilibrium probability of loan repayment for different levels of \( k \) is plotted as a function of the interest rate. The figure covers a broad range of real interest rate values (from negative 10 percent to positive 20 percent), encompassing the vast majority of realistic cases. From this picture it is easy to see how the response of a bank’s risk taking to a change in the interest rate depends on its capitalization. For low levels of \( k \), bank monitoring \( \hat{q} \) decreases with the interest rate \( r^* \), while the opposite happens at high levels.\(^{19}\)

When we allow the bank to change its target leverage ratio, an additional effect emerges and the ambiguity in the relationship between risk-taking and the reference interest rate is resolved. As the reference interest rate increases, so does the agency problem associated with limited liability. The bank’s response is to decrease its leverage ratio to limit the increase in the interest rate it has to pay on its uninsured liabilities. Fig. 2 describes this relationship. The equilibrium leverage ratio is plotted against the real interest rate. Note that, for illustrative purposes, the figure covers an extremely wide range of interest rates from minus 100 percent to plus 100 percent, which are well beyond what typically occurs in practice. At extremely low values of the reference interest

\(^{18}\) An equity premium of 6 percent is consistent with the historical average spread between US stock returns and risk-free interest rate as reported in Mehra and Prescott [40].

\(^{19}\) In our numerical example, the threshold value for \( k \) at which the relationship between the benchmark rate and bank risk taking reverses is about 0.55, which is a fairly high capitalization ratio in practice.
rate (below minus 15 percent), the agency problem is sufficiently small that the bank finds it optimal to be fully levered. For more realistic ranges of the interest rate, the model admits an internal solution and bank capital $k$ increases with the reference interest rate. However, the slope of this relationship is decreasing in the reference interest rate. When the real interest rate is zero (i.e., for $r^* = 1$), the optimal capital ratio is around 21%. Eventually, for unrealistically high levels of $r^*$, the relationship becomes flat once it hits its upper bound (this corresponds to the point where $\hat{q}(k) = 1$, see below).

Fig. 3 illustrates the relationship between the bank’s monitoring effort/probability of repayment and the reference interest rate for the case with endogenous leverage. For extremely low values of the real rate (exactly the values for which $\hat{k} = 0$), bank monitoring $\hat{q}$ is decreasing in
the reference interest rate. The intuition is straightforward. At these levels \( \hat{k} \) is in a corner (at zero) and does not move when the reference interest rate changes. It follows that the result related to a fixed capital structure applies. And since \( \hat{k} = 0 \), we obtain that \( \frac{d\hat{q}}{dr^*} = \frac{d\hat{q}}{dr^*}|_{k=0} < 0 \). For the most realistic range of the real rate, between minus 10 percent and plus 20 percent, \( \hat{q} \) admits an internal solution and is increasing in \( r^* \). Eventually, at a very high real interest rate (about 80 percent), \( \hat{q} \) hits its upper bound, which is exactly when the relationship between \( \hat{k} \) and \( r^* \) becomes flat.

As discussed in Section 5.1, as the portion of bank liabilities protected by insurance increases, the bank’s incentives to hold capital decrease. Indeed, in this model the only benefit for a bank from holding capital stems from the reduction in the rate it has to pay on risk sensitive deposits. Then, when the portion of insured liabilities crosses a certain threshold, risk sensitive ones are not sufficient to support a positive equilibrium level of capital (Fig. 4).

7. Some empirical evidence

In this section, we review and add to existing empirical evidence on the link between interest rates and bank risk taking. Broadly speaking, this evidence supports the key assumptions and the main empirical predictions from our model.

We start with a graphical presentation of simple correlations between the real interest rate and measures of bank leverage and risk taking using data from US banks. As measure of ex ante bank risk we use the weighted average internal risk rating assigned to loans by banks from the US Terms of Business Lending Survey, which is a quarterly survey on the terms of business lending of a stratified sample of about 400 banks conducted by the US Federal Reserve Bank. The survey asks participating banks about the terms of all commercial and industrial loans issued during the first full business week of the middle month in every quarter. We obtain quarterly data from the second quarter of 1997 until the third quarter of 2009 from the survey. The publicly available version of this survey encompasses an aggregate version of the terms of business lending, disaggregated by type of banks. Loan risk ratings vary from 1 to 5 and are increasing in risk. We use

![Equilibrium bank monitoring, \( \hat{q}(\hat{k}) \), as a function of the real interest rate \( r^* \).](image-url)
the weighted average risk rating score aggregate across all participating banks as a measure of bank risk.

As proxy for the real interest rate, we use the three-month average effective federal funds rate, adjusted for inflation by subtracting the three-month average change in the US consumer price index to obtain the real federal funds rate. The effective federal funds rate is a volume-weighted average of rates on trades arranged by major brokers and calculated daily by the Federal Reserve Bank of New York using data provided by the brokers. Since both the real federal funds rate and loan risk ratings are trended in our sample, we detrend both variables by deducting their linear time trend. We obtain similar results when detrending these variables using a Hodrick–Prescott filter.

Fig. 5 shows the relationship between the real federal funds rate and bank risk over the period 1997Q2–2009Q3. The data show a strong negative relationship between bank risk and the real policy interest rate that is statistically significant at the 1% level, consistent with a positive relationship between $\hat{q}$ and $r^*$ as predicted by the model when bank's set leverage optimally.

A key assumption underlying our prediction of an unambiguously negative relationship between bank risk and real policy rates is that bank leverage can be adjusted easily. With leverage fixed, however, poorly capitalized banks would monitor more following a drop in real rates, lowering bank risk taking. Therefore, when banks cannot adjust their capital ratio, the link between interest rates and bank risk taking is no longer universally negative, and can turn positive for poorly capitalized banks.

To test informally for this result, we consider a natural experiment. We assume that it is particularly costly for banks to adjust leverage during certain episodes, such as financial crises. In particular, we re-draw Fig. 5 for the periods before and after the start of the recent US financial crisis, which we date as the third quarter of 2007. It was during this quarter that Countrywide Financial and Bear Stearns started experiencing financial difficulties, two hedge funds of Bear Stearns with exposure to mortgage-backed securities were closed, and the Fed started to aggressively lower interest rates in response to growing signs of weakness in the US financial system. The idea of this sample split is that during this episode banks found it very difficult to issue
capital and thus adjust their leverage ratios. This was especially the case because the interbank markets froze and the supply of external capital for US banks became scarce and turned expensive in part due to heightened concerns about bank insolvencies and increased counterparty risk between financial institutions (see Acharya and Richardson [1] and Gorton [32]).

Fig. 6 portrays the link between interest rates and bank risk for the pre and post 2007Q3 periods, respectively. While this link is strongly negative throughout the period before 2007Q3, this negative relation breaks down during the crisis period, when in fact the correlation turns slightly positive. This is consistent with our model’s prediction that when bank leverage cannot be adjusted the link between interest rates and bank risk-taking need no longer be negative. In fact, the positive relationship during the crisis period (though based on a limited sample) is
consistent with the prediction that when banks are poorly capitalized and cannot adjust leverage (which surely was the case for many banks during the financial crisis, when banks faced a capital crunch) there is a positive relationship between interest rates and bank risk-taking.

In ongoing work, Dell’Ariccia et al. [23] use confidential data from the Federal Reserve’s Survey of Terms of Business Lending with information on individual bank responses to directly test the predictions from our model. Using loan level data, that paper confirms the results found here with aggregate figures. It controls for several bank-, regional-, and loan-specific characteristics and continues to find that there is a negative correlation between real interest rates and bank risk-taking, as measured by the real federal funds rate and the average internal risk rating assigned to loans by banks, respectively. Moreover, matching the lending survey with bank balance sheet data, it finds that this relation is more pronounced for banks with relatively high capital ratios, consistent with our model’s prediction. Also, the paper deals explicitly (by isolating the effect for small local banks) with the potential endogeneity of the Federal Fund rate and shows that the main predictions of our model are supported empirically.

Overall, this evidence lends support to the main empirical predictions from the model. These predictions are also consistent with a growing empirical literature covering a wide range of countries and periods, spanning from the US, to Europe, and Bolivia (e.g., [38,35,36,20]).

For example, Buch et al. [20], also using data from the Federal Reserve’s Survey of Terms of Business Lending, find that bank risk-taking increases following periods of monetary loosening for small domestic banks. This relationship breaks down for large domestic banks and foreign banks, arguably because the transmission channel is less strong for such banks as they are exposed more to shocks from abroad. Maddaloni and Peydro [38] compare data on Euro-area and US bank lending standards from the Bank Lending Survey (BLS) for the Euro-area countries and the Senior Loan Officer (SLO) survey for the US and find that low short-term interest rates soften standards for loans. Additionally, they find that this softening is amplified by securitization activity, weak supervision for bank capital, and low short term interest rates for an extended period. Similarly, Altunbas et al. [9] find evidence that “unusually” low interest rates over an extended period of time contributed to an increase in banks’ risk-taking in the US and Europe, as measured by expected default probabilities of individual banks. Ioannidou et al. [35] use the US federal funds rate as an instrument for monetary policy for Bolivia, which has a banking system that is almost fully dollarized, and find that a decrease in the US federal funds rate prior to loan origination raises the monthly probability of default on individual bank loans using data on individual loans from Bolivia’s credit register. Moreover, they find that this negative relation between interest rates and risk is more pronounced for banks with more liquid assets. To the extent that more liquid banks are less leveraged (after all, liquid assets can be used to reduce debt), this result is consistent with our model as well. Jimenez et al. [36], using detailed information on borrower quality from the Spanish credit registry, also find a positive association between low interest rates at loan origination and the probability of extending loans to borrowers with bad or no credit histories (i.e., risky borrowers).

8. Discussion and conclusions

This paper provides a theoretical foundation for the claim that a low interest rate environment may contribute to increases in bank risk taking. In our model, the net effect of a change in interest rate conditions (a change in a reference risk-free interest rate) on bank monitoring (an inverse measure of risk taking) depends on the balance of three forces: interest rate pass-through, risk shifting, and leverage. When banks can adjust their capital structures, a reduction in the reference
rate leads to greater leverage and lower monitoring. However, if a bank’s capital structure is instead fixed, the balance will depend on the degree of bank capitalization: when facing a drop in the reference rate, well capitalized banks will decrease monitoring. Highly levered banks, by contrast, may in fact increase monitoring and reduce risk as long the demand for loans they face is not overly convex. Further, the balance of these effects will depend on the structure and contestability of the banking industry, and is therefore likely to vary across countries and over time.

There are several potential extensions to our analysis that are useful to discuss. First, we focus on exogenous changes in the real yield on safe assets, and abstract from the policies that determine these interest rate movements, such as how central banks respond to the economic cycle and inflation pressures when choosing their policy stance, as well as possible motivations for the lifting of capital controls or for financial liberalizations that increase capital flows.

Another important simplifying assumption is that the cost of equity is independent from the bank’s leverage. Yet, our results would continue to hold in a more complex setting where the required return to equity is increasing in the degree of bank leverage. In this case, it is straightforward to see that, everything else equal, equilibrium leverage would be lower than in our base model since an increase in capitalization would have the additional benefit of reducing equity costs. Also, leverage would continue to be decreasing in the real interest rate, although the exact shape of this relationship would depend on the functional form assumed for the cost of equity as a function of leverage.

A third simplification in the paper is that we focus on credit risk and abstract from other important risks faced by banks, such as liquidity risk. While other frameworks may be better suited to study this issue (see, for example, [29] and [45]), our model could be adapted to capture risks on the liability side of the bank’s balance sheet. For instance, banks might choose to finance themselves through expensive long-term debt instruments or cheaper short-term deposits, which, however, carry a greater liquidity risk. In that context, the trade-off for a bank would be between a wider intermediation margin and a greater risk of failure should a liquidity run ensue. Hence, dynamics similar to those in this paper could be obtained. We leave all these extensions to future research.

The model has clear testable implications. First, in situations where banks are relatively unconstrained in raising capital and can adjust their capital structures, the model predicts a negative relationship between the risk-free interest rate and measures of bank risk. Second, in situations where banks face constraints, such as when their desired capital ratios are already below regulatory minimums for capital regulation, this negative relationship between risk-free interest rates and bank risk is less pronounced for poorly capitalized banks and in less competitive banking markets. Third, the model predicts a negative relationship between real interest rates and bank leverage. While we provide some simple empirical evidence in support of a negative relationship between the real interest rate and bank risk, and between the real interest rate and leverage, we leave more rigorous empirical analysis of these relationships to future research.

This paper’s findings have clear policy implications for central bankers and bank regulators and supervisors alike. At a general level, the model shows conditions under which real interest rates have a bearing on the riskiness of individual banks. At the same time, to the extent that

20 A growing literature focuses on funding liquidity risk of banks and the adverse liquidity spirals that such risk could generate in the event of negative shocks (see [27,19,3]) and on the role of monetary policy in altering bank fragility in the presence of liquidity risk ([2]; and [31]).
monetary policy, by altering short term interest rates, influences real interest rates, our model predicts a direct impact of monetary policy on bank risk and financial stability more generally.

Central banks should take these effects on financial stability into account when setting monetary policy, in addition to the traditional trade-off between employment and inflation, especially when they have a dual mandate that includes safeguarding financial stability. More specifically, our findings imply that the influence of changes in real interest rates on bank risk taking will differ across banks, depending on bank capitalization, deposit structure, and market structure. When leverage can easily be adjusted, bank risk taking will be higher during periods of low real interest rates. However, when banks are unable to adjust leverage, such as when they are capital constrained or during periods of financial stress, poorly capitalized banks will monitor more following a drop in real interest rates, lowering risk taking for such banks, albeit only if the banking sector is not highly competitive. Additionally, banks with a higher share of insured deposits unequivocally choose a higher leverage, as deposit insurance reduces the risk-sensitivity of deposits, reinforcing the link between interest rates and bank risk taking. These are important effects for bank supervisors to bear in mind when assessing the safety and soundness of banks. They also imply that supervisory attention may need to focus on certain types of banks, depending on the fraction of insured deposits they hold, their leverage, and their market positions.

The findings in this paper bear on the debate about how to integrate macro-prudential regulation into the monetary policy framework to meet the twin objectives of price and financial stability (see, for example, Blanchard, Dell’Ariccia, and Mauro [15]). Whether a trade-off between the two objectives emerges will depend on the type of shocks the economy is facing. For instance, no trade-off between price and financial stability may exist when an economy nears the peak of a cycle, when banks tend to take the most risks and prices are under pressure. Under these conditions, monetary tightening will decrease leverage and risk taking and, at the same time, contain price pressures. In contrast, a trade-off between the two objectives would emerge in an environment such as that in the run-up to the current crisis, with low inflation but excessive risk taking. Under these conditions, the policy rate cannot deal with both objectives at the same time: Tightening may reduce risk-taking, but will lead to an undesired contraction in aggregate activity and/or to deflation. Other (macroprudential) tools are then needed.

In particular, these effects point to the relevance of minimum capital requirements and their enforcement in the context of the twin objectives of price and financial stability. At times of low inflation but excessive risk taking, strict enforcement of minimum capital requirements could prevent banks from increasing leverage and contain risk taking resulting from a reduction in the policy rate. However, under regulatory capital forbearance, the effects of a monetary expansion aimed at stimulating economic activity could hurt financial stability. This is especially relevant during (or in the wake of) systemic banking crises. Under these conditions, a trade-off may emerge between the benefits of short-term forbearance (which could help weak, but solvent, banks to remain in operation and recover) and the need to limit the risk taking incentives associated with monetary policy easing.

The potential interaction between banking market conditions, monetary policy decisions, and bank risk-taking implied by our analysis can be seen as an argument in favor of the centralization of macro-prudential responsibilities within the monetary authority. And the complexity of this interaction points in the same direction. How these benefits balance with the potential for lower credibility and accountability associated with a more complex mandate and the consequent increased risk of political interference is a question for future research.
Appendix A

Proof of Proposition 1. Rational depositors demand an interest rate commensurate to the expected probability of repayment, \( r_D = \frac{r^*}{\hat{q}} \). Recall that, assuming an interior solution, we have \( \hat{q} = r_L - r_D(1 - k) \). Since in equilibrium depositors’ expectations must be correct, we can substitute for \( r_D \) as \( r_D = \frac{r^*}{\hat{q}} \) and rearrange to get

\[ q^2 - r_L q + r^*(1 - k) = 0. \]

Following Allen et al. (2010), we solve for \( q \) and take the larger root to obtain an equilibrium expression for bank monitoring:

\[ \hat{q}(k) = \frac{1}{2c} \left( r_L + \sqrt{r_L^2 - 4cr^*(1 - k)} \right). \] (16)

Again using the fact that, in equilibrium, \( r_D = \frac{r^*}{\hat{q}} \), we can rewrite the profit function as

\[ \Pi = \left( \hat{q}r_L - r^*(1 - k) - (r^* + \xi)k - \frac{1}{2}c\hat{q}^2 \right)L(\hat{r}_L). \]

The first order condition with respect to \( k \) is

\[ \frac{\partial \Pi}{\partial k} = \left( r^* - (r^* + \xi) + \frac{\partial \hat{q}}{\partial k} (\hat{r}_L - c\hat{q}) \right)L(\hat{r}_L) + \frac{\partial \Pi}{\partial \hat{r}_L} \frac{\partial \hat{r}_L}{\partial k} = 0. \]

The second term, \( \frac{\partial \Pi}{\partial \hat{r}_L} \frac{\partial \hat{r}_L}{\partial k} \), is zero by the envelope theorem, which implies a first order condition of

\[ -\xi + \frac{\partial \hat{q}}{\partial k} (\hat{r}_L - c\hat{q}) = 0. \] (17)

The second order condition can now be written as

\[ \frac{\partial^2 \Pi}{\partial k^2} = \frac{\partial L}{\partial \hat{r}_L} \left( r^* - (r^* + \xi) + \frac{\partial \hat{q}}{\partial k} (\hat{r}_L - c\hat{q}) \right) \]

\[ + L(\hat{r}_L) \left( \frac{\partial^2 \hat{q}}{\partial k^2} (\hat{r}_L - c\hat{q}) + \frac{\partial^2 \hat{q}}{\partial k \partial \hat{r}_L} (\hat{r}_L - c\hat{q}) \right). \]

The first term is zero from (17), leaving only

\[ \frac{\partial^2 \Pi}{\partial k^2} = \frac{\partial \hat{q}}{\partial k} \left( \frac{\partial \hat{r}_L}{\partial k} - c \frac{\partial \hat{q}}{\partial \hat{r}_L} \right) + \frac{\partial^2 \hat{q}}{\partial k^2} (\hat{r}_L - c\hat{q}). \] (18)

To sign this expression, we use the following auxiliary result.

Lemma 2. Around the optimal leverage ratio \( \hat{k} \), the optimal loan rate \( \hat{r}_L \) is increasing in \( k \):

\[ \left. \frac{\partial \hat{r}_L}{\partial k} \right|_{\hat{k}} > 0. \]

Proof. From the first order conditions with respect to \( r_L \) we have

\[ \frac{\partial \Pi}{\partial r_L} = qL(r_L) + \frac{\partial L(r_L)}{\partial r_L} \left( \hat{q}(r_L - r_D(1 - k)) - (r^* + \xi)k - \frac{1}{2}c\hat{q}^2 \right) + \frac{\partial \Pi}{\partial \hat{q}} \frac{\partial q}{\partial r_L} = 0. \]
Since the last term is zero by the envelope theorem, we can write:
\[
\hat{q}L(r_L) + \frac{\partial L(r_L)}{\partial r_L}\left(\hat{q}(r_L - r_D(1 - k)) - (r^* + \xi)k - \frac{1}{2}c\hat{q}^2\right) = 0.
\] (19)

Define \(Z \equiv \frac{\partial \Pi}{\partial r_L} = 0\). Then, using the Implicit Function Theorem we have \(\frac{\partial Z}{\partial k} = -\frac{\partial Z}{\partial r_L}\):
\[
\frac{\partial Z}{\partial r_L} = 2\hat{q}\frac{\partial L(r_L)}{\partial r_L} + L(r_L)\frac{\partial \hat{q}}{\partial r_L} + \frac{\partial^2 L(r_L)}{\partial r_L^2}\left(\hat{q}(r_L - r_D(1 - k)) - (r^* + \xi)k - \frac{1}{2}c\hat{q}^2\right)
+ \frac{\partial L(r_L)}{\partial r_L}\left(\hat{q}(r_L - r_D(1 - k)) - (r^* + \xi)k - \frac{1}{2}c\hat{q}^2\right)\frac{\partial \hat{q}}{\partial r_L}.
\]

The last term is zero from the envelope theorem, leaving,
\[
\frac{\partial Z}{\partial r_L} = 2\hat{q}\frac{\partial L(r_L)}{\partial r_L} + L(r_L)\frac{\partial \hat{q}}{\partial r_L} + \frac{\partial^2 L(r_L)}{\partial r_L^2}\left(\hat{q}(r_L - r_D(1 - k)) - (r^* + \xi)k - \frac{1}{2}c\hat{q}^2\right).
\]

This expression is the second order condition for maximization of bank profits, \(\Pi\), with respect to the loan rate \(r_L\). Assume for now that it is satisfied, so that \(\frac{\partial^2 \Pi}{\partial r_L^2} < 0\). It is straightforward to show that for a broad class of loan demand functions, \(L(r_L)\), such as for linear demand, the second order condition \(\frac{\partial^2 \Pi}{\partial r_L^2} < 0\) will be satisfied. We discuss below the case where \(\frac{\partial^2 \Pi}{\partial r_L^2} < 0\) is not satisfied.

Now, to compute \(\frac{\partial Z}{\partial k}\), we first write \(Z\) in a way that reflects the equilibrium condition that \(r_D = r_L^*\), since \(r_D\) is determined after \(k\) and \(r^*\) are chosen:
\[
Z = \hat{q}L(r_L) + \frac{\partial L(r_L)}{\partial r_L}\left(\hat{q}r_L - r^*(1 - k) - (r^* + \xi)k - \frac{1}{2}c\hat{q}^2\right) = 0.
\]

We can now differentiate this to obtain
\[
\frac{\partial Z}{\partial k} = \frac{\partial \hat{q}}{\partial k}L(r_L) + \frac{\partial L(r_L)}{\partial r_L}(r^* - (r^* + \xi)) + \frac{\partial L(r_L)}{\partial r_L}(r_L - c\hat{q})\frac{\partial \hat{q}}{\partial k}
= \frac{\partial \hat{q}}{\partial k}L(r_L) + \frac{\partial L(r_L)}{\partial r_L}\left(-\xi + (r_L - c\hat{q})\frac{\partial \hat{q}}{\partial k}\right).
\]

However, from (17), the FOC with respect to \(k\), we know that the term in brackets is zero. This means that, for \(\hat{q}(k) = \frac{1}{2r}(r_L + \sqrt{r_L^2 - 4cr^*(1 - k)})\),
\[
\frac{\partial Z}{\partial k} = \frac{\partial \hat{q}}{\partial k}L(r_L) = \frac{L(r_L)r^*}{\sqrt{r_L^2 - 4cr^*(1 - k)}} > 0.
\]

Thus, \(\frac{\partial Z}{\partial k} = -\frac{\partial \hat{q}}{\partial k} > 0\), as desired. \(\Box\)

We can now use Lemma 2 to establish that, around the equilibrium value of capital \(\hat{k}\), \(\frac{\partial \hat{q}}{\partial k} > 0\). From this, it also follows that \(\frac{\partial \hat{q}}{\partial k} > 0\). We therefore need to sign \((\frac{\partial \hat{q}}{\partial k} - c\frac{\partial \hat{q}}{\partial k})\). From (16), we can write
\[
c\frac{\partial \hat{q}}{\partial k} = \frac{1}{2}\frac{\partial \hat{r}_L}{\partial k} + \frac{cr^* + \frac{1}{2}\frac{\partial \hat{r}_L}{\partial k}r_L}{\sqrt{r_L^2 - 4cr^*(1 - k)}}.
\]
Thus
\[
\frac{\partial \hat{r}_L}{\partial k} - c \frac{\partial \hat{q}}{\partial k} = \frac{1}{2} \frac{\partial \hat{r}_L}{\partial k} \left( 1 - \frac{\hat{r}_L}{\sqrt{r^*_L - 4cr^*(1 - k)}} \right) - \frac{cr^*}{\sqrt{r^*_L - 4cr^*(1 - k)}},
\]
which is negative because \( \hat{r}_L \geq \sqrt{r^*_L - 4cr^*(1 - k)} \) for any \( k \leq 1 \). Note as well that
\[
\frac{\partial^2 \hat{q}}{\partial k^2} = \frac{1}{2c} \left( r_L + \sqrt{r^*_L - 4cr^*(1 - k)} \right) = -2c \frac{(r^*)^2}{(r^*_L - 4cr^* + 4ckr^*)^3} < 0.
\]
It follows that profits are concave in \( k \).

Define now \( G = \frac{\partial \Pi}{\partial k} = 0 \) and \( H = \frac{\partial^2 \Pi}{\partial k^2} < 0 \). Using the Implicit Function Theorem, we then have \( \frac{d k}{d r^*} = -\frac{\hat{G}}{\hat{r}^*} \). Since the denominator is negative, the sign of \( \frac{d k}{d r^*} \) will be the same as that of \( \frac{\hat{G}}{\hat{r}^*} \). The numerator is
\[
\frac{\partial G}{\partial r^*} = \frac{\partial (-\xi + \frac{\hat{q}}{\hat{r}_L} (\hat{r}_L - c\hat{q}))}{\partial r^*} = \frac{\partial \hat{q}}{\partial k} \left( \frac{\partial \hat{r}_L}{\partial r^*} - c \frac{\partial \hat{q}}{\partial r^*} \right) + \hat{r}_L - c\hat{q} \frac{\partial^2 \hat{q}}{\partial k \partial r^*}.
\]
The first term is positive since \( \frac{\partial \hat{q}}{\partial k} > 0 \), \( \frac{\partial \hat{r}_L}{\partial r^*} > 0 \), and \( \frac{\partial \hat{q}}{\partial r^*} < 0 \). The second term depends on the sign of \( \frac{\partial^2 \hat{q}}{\partial k \partial r^*} \), which is given by
\[
\frac{\partial^2}{\partial k \partial r^*} \left( \frac{1}{2c} \left( r_L + \sqrt{r^*_L - 4cr^*(1 - k)} \right) \right) = \frac{r^*_L - 2cr^*(1 - k)}{(r^*_L - 4cr^*(1 - k))^2} > 0.
\]
It follows that \( \frac{d k}{d r^*} > 0 \), as desired, as long as the second order condition with respect to the loan rate is satisfied, i.e., \( \frac{\partial^2 \Pi}{\partial \hat{r}_L^2} < 0 \).

Finally, suppose instead that the second order condition is not satisfied, meaning that, at the value of \( r_L \) satisfying the first order condition for profit maximization, we have \( \frac{\partial^2 \Pi}{\partial \hat{r}_L^2} = \frac{\partial Z}{\partial r_L} > 0 \). This implies that either there is no solution, with \( r_L \to \infty \) (note that \( r_L \to 0 \) cannot be a solution since it would lead to negative profits), or there is a corner solution in the sense that the demand curve intersects the vertical axis at some finite loan rate denoted by \( R \), and the bank finds it optimal to charge this maximal loan rate: \( r_L = R \).

Since the former case where there is no solution is uninteresting, we focus on the latter case where the bank maximizes profit by charging the maximal loan rate \( r_L = R \). Note that demand becomes zero for \( r_L > R \), so that an increase in \( r^* \) cannot raise the loan rate further. Moreover, were the loan rate to fall following an increase in \( r^* \), \( r_L \) would then be interior, contradicting the result in Lemma 2. Therefore, an increase in \( r^* \) must leave the loan rate unchanged and equal to the maximal rate \( R \).

The analysis of this case is now straightforward. Since, at least locally, the loan rate remains equal to the maximal rate \( R \), the bank’s maximization problem becomes:
\[
\max_k \Pi = L \left( \hat{q}R - r - k\xi - \frac{c\hat{q}^2}{2} \right).
\]
This gives the first order condition
\[- \frac{r^*}{2} - \xi + \frac{r^* R}{2\sqrt{R^2 - 4cr^*(1 - k)}} = 0,\]
with solution
\[\hat{k} = 1 - R^2 \frac{\xi(r^* + \xi)}{cr^*(r^* + 2\xi)^2}.\]  
(21)

It is now immediate that \(\frac{d\hat{k}}{dr^*} > 0\), as desired. Therefore, whenever a solution exists, we have that \(\frac{d\hat{k}}{dr^*} > 0\), as desired. □

**Proof of Lemma 1.** We can write \(\frac{d\hat{r}_L}{dr^*} = \frac{\partial \hat{r}_L}{\partial k}(\hat{k}_L | \hat{k}) + \frac{\partial \hat{r}_L}{\partial r^*}(\hat{k}_L | \hat{k})\), where the notation \(\frac{\partial \hat{r}_L}{\partial k}(\hat{k}_L | \hat{k})\) refers to the derivative of the equilibrium loan rate with respect to the real interest rate \(r^*\), for a given fixed capital ratio \(k\). As above, \(\frac{\partial \hat{r}_L}{\partial k}(\hat{k}_L | \hat{k})\) is the derivative of the loan rate around the equilibrium level of capital, \(\hat{k}\). Therefore, we have that the first term, \(\frac{\partial \hat{r}_L}{\partial k}(\hat{k}_L | \hat{k})\), is positive from Lemma 2 and Proposition 1. Therefore, the only remaining term to sign is \(\frac{\partial \hat{r}_L}{\partial r^*}(\hat{k}_L | \hat{k})\). For this, recall again the first order condition for profit maximization with respect to \(r_L\) obtained in (19):
\[\frac{\partial \Pi}{\partial r_L} = \hat{q}L(r_L) + \frac{\partial L(r_L)}{\partial r_L}(\hat{q}(r_L - r_D(1 - k)) - (r^* + \xi)k - \frac{1}{2}c\hat{q}^2) = 0.\]

We again define \(Z \equiv \frac{\partial \Pi}{\partial r_L} = 0\). Then, using the Implicit Function Theorem we have \(\frac{dr_L}{dr^*} = -\frac{\partial Z}{\partial r^*}\). For an interior solution for the loan rate, the denominator must be negative. For the numerator, we have
\[\frac{\partial Z}{\partial r^*} = \frac{\partial \hat{q}}{\partial r^*}L(r_L) + \frac{\partial L(r_L)}{\partial r_L}(r_L - c\hat{q})\frac{\partial q}{\partial r^*} = \frac{\partial \hat{q}}{\partial r^*}L(r_L) - \frac{\partial L(r_L)}{\partial r_L}(1 - (r_L - c\hat{q})\frac{\partial q}{\partial r^*}).\]

Now, using the fact that \(\hat{q} = \frac{1}{2c}(r_L + \sqrt{r_L^2 - 4cr^*(1 - k)})\), we know that
\[\frac{\partial \hat{q}}{\partial r^*} = -\frac{1 - k}{\sqrt{r_L^2 - 4cr^*(1 - k)}}.\]

For ease of exposition, let us define \(W = \sqrt{r_L^2 - 4cr^*(1 - k)}\). We can substitute this into \(\frac{\partial Z}{\partial r^*}\) to obtain
\[\frac{\partial Z}{\partial r^*} = -\frac{1 - k}{W}L(r_L) - \frac{\partial L(r_L)}{\partial r_L}(1 - (r_L - c\frac{1}{2c}(r_L + W))\left( - \frac{1 - k}{W} \right)).\]

We can rewrite \(Z = 0\) as
\[L(r_L) = -\frac{\frac{\partial L(r_L)}{\partial r_L}(\hat{q}r_L - r^*(1 - k) - (r^* + \xi)k - \frac{1}{2}c\hat{q}^2)}{\hat{q}} = -\frac{\frac{\partial L(r_L)}{\partial r_L}(\hat{q}r_L - r^* - k\xi - \frac{1}{2}c\hat{q}^2)}{\hat{q}}.\]
and we can substitute into the above
\[
\frac{\partial Z}{\partial r} = \frac{\partial L(r_L)}{\partial r_L} \left( 1 - k \left( \frac{\hat{q} r_L - r^*}{W} - k \xi - \frac{1}{2} c \hat{q}^2 \right) \right) - \left( 1 - \left( r_L - c \frac{1}{2c} (r_L + W) \right) \left( \frac{1 - k}{W} \right) \right).
\]
Substituting now for \( \hat{q} \) and simplifying yields
\[
\frac{\partial Z}{\partial r} = \frac{\partial L(r_L)}{\partial r_L} \left( - \frac{1}{4 r^* H} (r^* (r_L + W) + 2 k \xi (r_L - W) + k r^* (r_L + W)) \right).
\]
From the equilibrium solution for \( \hat{q} \), we know that
\[
2 c \hat{q} = r_L + \sqrt{r_L^2 - 4 c r^*(1 - k)} = r_L + W.
\]
This allows us to write
\[
\frac{\partial Z}{\partial r} = \frac{\partial L(r_L)}{\partial r_L} \left( - \frac{1}{4 r^* H} \left( r^* (r_L + W) + 2 k \xi (r_L - W) + k r^* (r_L + W) \right) \right).
\]
It must also be that
\[
2 (r_L - c \hat{q}) = 2 r_L - \left( r_L + \sqrt{r_L^2 - 4 c r^*(1 - k)} \right) = r_L - \sqrt{r_L^2 - 4 c r^*(1 - k)}.
\]
This term shows up in the expression above for \( \frac{\partial Z}{\partial r^*} \). We can therefore substitute this back into \( \frac{\partial Z}{\partial r^*} \) to obtain
\[
\frac{\partial Z}{\partial r^*} = - \frac{\partial L(r_L)}{\partial r_L} \left( - \frac{1}{4 r^* H} \left( r^* (2 c \hat{q} + 2 k \xi (2 (r_L - c \hat{q}) + k r^* 2 c \hat{q})) \right) \right).
\]
This has to be satisfied as an identity in equilibrium: \( \frac{\partial Z}{\partial r^*} = 0 \) for any value of \( r^* \) at the equilibrium choice of \( k \).

Proof of Proposition 2. From the proof of Proposition 1, we have that since \( r_D = \frac{r^*}{q} \), we can rewrite the bank’s profits as
\[
\Pi = \left( \frac{\hat{q} r_L - r^* (1 - k) - (r^* + \xi) k - \frac{1}{2} c \hat{q}^2}{L} \right) L(r_L).
\]
The first order condition with respect to \( k \) is
\[
\frac{\partial \Pi}{\partial k} = r^* - (r^* + \xi) + \frac{\partial \hat{q}}{\partial k} (r_L - c \hat{q}) = -\xi + \frac{\partial \hat{q}}{\partial k} (r_L - c \hat{q}) = 0.
\]
This has to be satisfied as an identity in equilibrium: \( \frac{\partial \Pi}{\partial k} = 0 \) for any value of \( r^* \) at the equilibrium choice of \( k \).
Now consider the following derivative:

\[
\frac{d}{dr^*} \left( \frac{\partial \Pi}{\partial k} \right) = \frac{\partial}{\partial r^*} \left( -\xi + \frac{\partial \hat{q}}{\partial k} (\hat{r}_L - c\hat{q}) \right) = \frac{\partial \hat{q}}{\partial k} \left( \frac{d\hat{r}_L}{dr^*} - c \frac{d\hat{q}}{dr^*} \right) + \frac{\partial q^2}{\partial k \partial r^*} (\hat{r}_L - c\hat{q}).
\]

Given that \( \frac{\partial \Pi}{\partial k} \) is identically equal to zero, this expression must also equal zero:

\[
\frac{d}{dr^*} \left( \frac{\partial \Pi}{\partial k} \right) = 0 \iff \frac{\partial \hat{q}}{\partial k} \left( \frac{d\hat{r}_L}{dr^*} - c \frac{d\hat{q}}{dr^*} \right) + \frac{\partial q^2}{\partial k \partial r^*} (\hat{r}_L - c\hat{q}) = 0.
\]

We can compute

\[
\frac{\partial q^2}{\partial k \partial r^*} = \frac{\partial^2}{\partial k \partial r^*} \left( \frac{1}{2c} \left( r_L + \sqrt{r_L^2 - 4cr^*(1-k)} \right) \right) = \frac{r_L^2 - 2cr^*(1-k)}{(r_L^2 - 4cr^*(1-k))^2} > 0.
\]

We know already that \( \frac{d\hat{q}}{dr^*} > 0 \), and that \( \hat{r}_L - c\hat{q} \geq 0 \). Therefore, the only way for the equilibrium condition \( \frac{d}{dr^*} \left( \frac{\partial \Pi}{\partial k} \right) = 0 \) to be satisfied is if \( \frac{d\hat{r}_L}{dr^*} - c \frac{d\hat{q}}{dr^*} < 0 \). However, since (23) only holds around the equilibrium value of capital, \( \hat{k} \), we can apply Lemma 1 to sign \( \frac{d\hat{r}_L}{dr^*} \) as positive. It then follows that \( \frac{d\hat{q}}{dr^*} > 0 \).

Finally, for the case where the second order condition for profit maximization with respect to the loan rate is not satisfied, the result can be obtained more directly. From the analysis in the proof of Proposition 1, we have that when \( r_L = R \), the maximal loan rate, the optimal level of bank capital is given by \( \hat{k} = 1 - \frac{R^2}{2} \frac{\xi (r^* + \xi)}{cr^* (r^* + 2\xi)^2} \). We can substitute the solution \( \hat{k} \) back into the formula for \( \hat{q} \) to obtain

\[
\hat{q} = R \frac{(r + \xi)}{c(r + 2\xi)}.
\]

It is now immediate that \( \frac{d\hat{q}}{dr^*} > 0 \) for this case as well. \( \square \)

Proof of Proposition 3. As before, rational depositors demand an interest rate commensurate to the expected probability of repayment, \( r_D = \frac{r^*}{E[q]} \). Assuming an interior solution, we have \( \hat{q} = \frac{r_L - r_D(1-k)}{c} \). Since in equilibrium depositors’ expectations must be correct, we can substitute for \( r_D \) as \( r_D = \frac{r_L}{E[q]} \) and solve for equilibrium monitoring \( q \) to obtain

\[
\hat{q}(k) = \frac{1}{2c} \left( r_L + \sqrt{r_L^2 - 4cr^*(1-k)} \right).
\]

This implies

\[
\frac{d\hat{q}(k)}{dr} \bigg|_{k} = \frac{1}{2c} \left( \frac{dr_L}{dr} \right) \bigg|_{k} + \frac{-2c(1-k) + r_L \frac{dr_L}{dr} |_k}{\sqrt{r_L^2 - 4cr^*(1-k)}}.
\]

The lending rate is obtained from the maximization of the bank’s profit, and is determined by the following FOC:

\[
\frac{\partial \Pi}{\partial r_L} = L \left( \frac{r_L - r_D(1-k)}{c} \right) + \frac{\partial L}{\partial r_L} \left( c \left( \frac{r_L - r_D(1-k)}{c} \right)^2 - k(r^* + \xi) \right) = 0.
\]
This expression defines the optimal loan rate \( r_L \), assuming the second order condition is satisfied. As before, define \( Z = \frac{\partial r}{\partial r_L} = 0 \), and note that, from the Implicit Function Theorem, \( \frac{dr}{dr_L} = -\frac{\partial Z}{\partial r_L} \).

The denominator is simply the second derivative of bank profits with respect to the loan rate, and is negative if the second order condition is satisfied so that the solution for the optimal loan rate is interior. We can calculate the numerator as

\[
\frac{\partial Z}{\partial r} = \frac{\partial}{\partial r} \left( \left( \frac{r_L - r_D(1-k)}{c} \right) + \frac{\partial L}{\partial r_L} \left( \frac{r_L - r_D(1-k)}{c} \right)^2 - k(r^* + \xi) \right)
\]

\[
= -\frac{\partial r_D}{\partial r} \left( \frac{r_L - r_D(1-k)}{c} \right) + \frac{\partial L}{\partial r_L} \left( \frac{r_L - r_D(1-k)}{c} \right)^2 - \frac{\partial L}{\partial r_L} k.
\]

For \( k = 1 \), we have \( \frac{\partial Z}{\partial r_L} |_{k=1} = -\frac{\partial r_D}{\partial r} k > 0 \). From the Implicit Function Theorem, this implies that \( \frac{dr}{dr_L} > 0 \), and therefore that \( \frac{dq}{dr} > 0 \) for \( k = 1 \). Since \( r_L \) is continuous in \( k \), and \( q \) is continuous in \( r_L, k \), and \( r^* \), this implies that \( \frac{dq}{dr} > 0 \) for \( k \) sufficiently large.

**Proof of Corollary 1.** If loan demand is linear, so that \( L = A - br_L \), then \( \frac{d^2 L}{dr_L^2} = 0 \), and we can use the Implicit Function Theorem as in the proof of Proposition 3 to show that \( \frac{dr}{dr_L} = \frac{1}{4} \frac{dr_D}{dr} \) at \( k = 0 \). And since \( r_D = \frac{r}{q} \),

\[
\frac{dr_D}{dr_L} = \frac{1}{3} \left( \frac{q - r^* \frac{dq}{dr}}{q^2} \right).
\]

Plugging this into (27), we get

\[
\frac{d\hat{q}(k)}{dr} = \frac{1}{2c} \left( \frac{1}{3} \left( \frac{q - r^* \frac{dq}{dr}}{q^2} \right) + -2c + r_L \frac{1}{3} \left( \frac{q - r^* \frac{dq}{dr}}{q^2} \right) \right),
\]

which solving for \( \frac{d\hat{q}(k)}{dr} \) yields:

\[
\frac{d\hat{q}(k)}{dr} = \frac{\hat{q}(r_L + \sqrt{r_L^2 - 4cr^*} - 6c\hat{q})}{r^*(r_L + \sqrt{r_L^2 - 4cr^*}) + 6q^2c\sqrt{r_L^2 - 4cr^*}}.
\]

The denominator of (28) is positive, and remembering that at \( k = 0 \),

\[
\hat{q}(k) = \frac{1}{2c} \left( r_L + \sqrt{r_L^2 - 4cr^*} \right),
\]

we can write the numerator of (28) as

\[
\hat{q}(2c\hat{q} - 6c\hat{q}) = -4\hat{q}^2 < 0.
\]

This tells us that \( \frac{d\hat{q}(k)}{dr} < 0 \) at \( k = 0 \). By continuity, this must also be true for \( k \) strictly positive. From Proposition 3 we know, however, that \( \frac{d\hat{q}(k)}{dr} > 0 \) for \( k = 1 \). This establishes the result.

**Proof of Proposition 4.** Start from \( \hat{q} = \frac{r_L + \sqrt{r_L^2 - 4cr^*}}{2c} \). This imposes a lower bound on \( r_L \):

\[
r_L \geq r_L = \sqrt{4cr^*(1-k)}.\]

Using the expression for \( q = \frac{r_L - r_D(1-k)}{c} \), we can write the bank’s expected profits as
\[ \Pi = \left( \frac{1}{2} \hat{\gamma}^2 c - (r^* + \xi) k \right) L(\hat{r}_L). \] (29)

From \( \hat{\gamma} = \frac{r_L + \sqrt{r_L^2 - 4cr^*(1-k)}}{2c} \), we have that, when \( r_L \) is at its lower bound, \( \hat{\gamma}(r_L) = \sqrt{\frac{r^*(1-k)}{c}} \), which, plugged into (29) gives

\[ \Pi(k) \bigg|_{r_L} = \left( \frac{r^*(1-k)}{2} - (r^* + \xi) k \right) L(\hat{r}_L). \]

We can now impose the zero profit condition and solve for \( k \), we get

\[ k \geq k = \frac{r^*}{3r^* + 2\xi}. \]

Note that, for \( k < k \), bank profits would be positive since a loan rate below \( r_L \) is not possible.

Now consider the zero profit condition for a given \( k \geq k \):

\[ \hat{\Pi} = L \left( \hat{\gamma} r_L - r^* - k\xi - \frac{c \gamma^2}{2} \right) = \left( \frac{1}{4c} \left( r_L^2 + r_L \sqrt{r_L^2 - 4cr^*(1-k)} - 2cr^*(1-k) \right) - (r^* + \xi) k \right) = 0 \] (30)

after substituting for \( \hat{\gamma} = \frac{r_L + \sqrt{r_L^2 - 4cr^*(1-k)}}{2c} \). At \( k = 1 \), (30) becomes

\[ \frac{1}{2c} r_L^2 - (r^* + \xi) = 0, \]

which gives \( r_L = \sqrt{(r^* + \xi)2c} \). We can now plug this into \( \hat{\gamma}|_{k=1} = \frac{r_L}{c} \) to get \( \hat{\gamma} = \frac{\sqrt{(r^* + \xi)2c}}{c} \).

From here we obtain

\[ \frac{d\hat{\gamma}}{dr^*} = \frac{1}{2\sqrt{\frac{1}{2}c(r^* + \xi)}} = \frac{1}{\sqrt{2c(r^* + \xi)}} > 0. \]

Now consider the opposite extreme: \( k = k = \frac{r^*}{3r^* + 2\xi} \). By design, \( r_L = \sqrt{4cr^*(1-k)} \) so that \( \hat{\gamma}(r_L) = \sqrt{\frac{r^*(1-k)}{c}} = \sqrt{2r^*rac{\xi + r^*}{c(2\xi + 3r^*)}} \). Differentiating,

\[ \frac{\partial \hat{\gamma}}{\partial r^*} = \frac{1}{2c(3r^* + 2\xi)} \frac{\sqrt{2} \left( r^* \right)^2 + 4r^*\xi + 2\xi^2}{\sqrt{\frac{1}{c} \left( 3r^* + 2\xi \right)^2 (r^* + \xi)}} > 0, \]

as desired. \( \square \)

**Proof of Proposition 5.** After substituting in \( \hat{\gamma} = \frac{r_L + \sqrt{r_L^2 - 4cr^*(1-k)}}{2c} \), maximizing profits

\[ \max_k \Pi = L \left( \hat{\gamma} r_L - r^*(1-k) - (r^* + \xi) k - \frac{c \hat{\gamma}^2}{2} \right) \]

gives the first order condition

\[ \frac{\partial \Pi}{\partial k} = -\frac{r^*}{2} - \xi + \frac{r^* r_L}{2\sqrt{r_L^2 - 4cr^*(1-k)}} = 0. \]
We can solve this to obtain
\[ \hat{k} = 1 - r^2 L \xi (r^* + \xi)(r^* + 2\xi)^2. \]

We now impose zero profits to obtain the lending rate
\[ \hat{r}_L = \sqrt{\frac{2cr^*(r^* + 2\xi)^2}{3r^*\xi + r^* + 2\xi^2}}. \]

Plugging back into \( \hat{k} \) yields
\[ \hat{k} = \frac{r^*\xi + r^*}{3r^*\xi + r^* + 2\xi^2}. \] (31)

From (31) we obtain
\[ \frac{d\hat{k}}{dr^*} = \frac{\xi(4\xi^3 + 10r^*\xi^2 + 2r^* + 8r^*2\xi)}{(r^* + 4\xi^3 + 8r^*\xi^2 + 5r^*2\xi)(3r^*\xi + r^* + 2\xi^2)} > 0. \]

We can also write
\[ \hat{q} = \sqrt{\frac{r^*4(r^* + \xi)^2}{2c(3r^*\xi + r^* + 2\xi^2)}}, \]

from which it is immediate that there always exists a \( c \) large enough that \( \hat{q} < 0 \). More precisely,
\[ \frac{r^*4(r + \xi)^2}{2c(3r^*\xi + r^* + 2\xi^2)} < 1 \iff r^*4(r^* + \xi)^2 < 2c(3r^*\xi + r^* + 2\xi^2) \]
\[ \iff \frac{2r^*(r^* + \xi)^2}{3r^*\xi + r^* + 2\xi^2} < c. \]

Now note that
\[ \frac{d\hat{q}}{dr^*} = \frac{(4r^*\xi + r^* + 2\xi^2)}{\sqrt{2c(\xi + r^*)(r^* + 2\xi)^2}} > 0, \]

as desired. \( \square \)

References


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