

Building Brand Awareness in Dynamic Oligopoly Markets

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Companies spend hundreds of millions of dollars annually on advertising to build and maintain awareness for their brands in competitive markets. However, awareness formation models in the marketing literature ignore the role of competition. Consequently, we lack both the empirical knowledge and normative understanding of building brand awareness in dynamic oligopoly markets. To address this gap, we propose an N -brand awareness formation model, design an extended Kalman filter to estimate the proposed model using market data for five car brands over time, and derive the optimal closed-loop Nash equilibrium strategies for every brand. The empirical results furnish strong support for the proposed model in terms of both goodness-of-fit in the estimation sample and cross-validation in the out-of-sample data. In addition, the estimation method offers managers a systematic way to estimate ad effectiveness and forecast awareness levels for their particular brands as well as competitors' brands. Finally, the normative analysis reveals an inverse allocation principle that suggests—contrary to the proportional-to-sales or competitive parity heuristics—that large (small) brands should invest in advertising proportionally less (more) than small (large) brands.

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1. Introduction

The real assets of companies are often intangible, such as brand names, rather than physical, such as plant and machinery. For example, the Ford Motor Company recently sold physical assets and invested over 12 billion dollars to acquire prestigious brand names: Jaguar, Aston Martin, Volvo, and Land Rover. As reported by *Forbes*,

None of these marquees brought much in the way of plant and equipment but plant and equipment aren't what the new business model is about. It's about brands and brand building... Ford has been selling things you can touch and buying what exists only in the consumers' minds. (Akasie 2000, p. 30)

To maintain and enhance intangible assets, managers spend hundreds of millions of dollars annually on advertising to build brand awareness, which is a major component of brand equity (Keller 2002, p. 67) and it can be measured (e.g., Interbrand rankings of 100 global brands). But competing brands also seek to increase brand awareness; hence managers must take into account the presence of multiple competitors in determining their best course of action. However, extant awareness formation models in marketing have

ignored the role of competition (see Mahajan et al. 1984, Naik et al. 1998, Bass et al. 2007). Consequently, the literature contains sparse empirical knowledge on the effectiveness of own and competitors' advertising in building awareness for mature products; it also lacks normative guidelines for the optimal strategies to pursue for enhancing brand awareness.

To address these issues, we investigate awareness formation in dynamic markets with multiple competitors. In §3, we propose an N -brand awareness formation model, which extends the models of Sethi (1983) and Sorger (1989) by not only allowing competition across $N > 2$ brands, but also incorporating market expansion and brand confusion effects. We empirically validate the model using market data on awareness of five car brands over time. Because this model is a simultaneous system of coupled nonlinear difference equations, we have to estimate its parameters by applying the extended Kalman filter (e.g., Xie et al. 1997). The empirical results furnish strong support for the proposed model in terms of not only in-sample fit, but also out-of-sample forecasts. In addition, managers can use the estimation approach to estimate ad effectiveness and forecast awareness levels for their product markets; §4.2 describes the approach.

Section 5 derives the closed-loop Nash equilibrium strategies for every brand in a dynamic oligopoly. This normative analysis reveals an inverse allocation principle, which indicates that large (small) brands should invest in advertising proportionally less (more) than small (large) brands. The novelty of this principle is the proof we furnish that inverse allocation is optimal, which is contrary to the textbook recommendations based on proportional-to-sales or competitive parity heuristics. Finally, §6 concludes by suggesting avenues for further research. But first, to identify gaps in empirical and normative understanding, we review the marketing and management science literatures.

2. Literature Review

We describe the extant findings from both awareness formation and differential game models.

2.1. Awareness Formation Models

Awareness formation models describe the growth and decay of a brand's awareness over time in response to advertising efforts. Mahajan et al. (1984) reviewed the marketing science literature on awareness formation models. Here we present models developed by Blattberg and Golanty (1978) and Dodson and Muller (1978), whose key features we incorporate in our formulation.

Blattberg and Golanty (1978) develop a model called TRACKER, where change in brand awareness is driven by advertising effort that influences the unaware segment of the market. Denoting the fraction of an aware segment by A_t and advertising effort by u_t (operationalized by gross rating points (GRPs)), the model is expressed as $\Delta A_t = A_t - A_{t-1} = f(u_t) \cdot (1 - A_{t-1})$, where the concave response function $f(u)$ captures diminishing returns.

Dodson and Muller (1978) extend the TRACKER specification by incorporating word-of-mouth and forgetting effects. They specify awareness change as $\Delta A_t = f(u_t)(1 - A_{t-1}) + \beta A_{t-1}(1 - A_{t-1}) - \delta A_{t-1}$. Specifically, the term $A_{t-1}(1 - A_{t-1})$ captures the word-of-mouth effect because of the social interaction between the aware segment A_t and the unaware market $(1 - A_{t-1})$. The last term $(-\delta A_{t-1})$ reflects the loss of awareness due to forgetting effects.

Overall, the common feature across these models, including the recent ones (e.g., Naik et al. 1998, Bass et al. 2007), is the absence of competitive brands and their strategic responses over time. Because differential game models shed light on dynamic competitive strategies, we next review this management science literature.

2.2. Differential Game Models

Differential game models facilitate the study of market dynamics by applying differential equations and

game-theoretic concepts to obtain normative solutions to managers' decisions. For surveys of this literature, see Feichtinger et al. (1994), Dockner et al. (2000), Sethi and Thompson (2000), Erickson (2003), and Jorgensen and Zaccour (2004). Most studies in this genre analyze duopoly markets (e.g., Erickson 1992, Chintagunta and Vilcassim 1992, Chintagunta and Jain 1995, Fruchter and Kalish 1997, Prasad and Sethi 2004), while a few notable studies on oligopoly markets include Teng and Thompson (1983), Fershtman (1984), Dockner and Jorgensen (1992), Erickson (1995), Fruchter (1999), and Naik et al. (2005). In these oligopoly models, closed-form solutions for optimal decisions are rarely available. In contrast, we will provide closed-form results in oligopoly markets by extending the Sethi (1983) model.

The dynamics in Sethi (1983) can be expressed as

$$dA(t)/dt = \rho u(t)\sqrt{1 - A(t)} - \delta A(t), \quad A(0) = A_0, \quad (1)$$

where $A(t)$ represents awareness at time t , and ρ and δ denote ad effectiveness and decay constant, respectively. Similarities between Sethi (1983) and Dodson and Muller (1978) are as follows. Specifically, both models consider monopoly markets; the awareness change, dA/dt , in continuous time is analogous to ΔA_t in the discrete-time version; the last term $(-\delta A)$ characterizes the decay in awareness due to forgetting effects. Furthermore, to provide an interpretation of the square root in (1), Sorger (1989) and Erickson (2003, p. 24) observe that $\sqrt{1 - A} \approx (1 - A) + kA(1 - A)$, where the best-fitting value of $k = 13/14$ for $A \in [0, 1]$ (as noted by an anonymous reviewer); the first term $(1 - A)$ represents the unaware segment, and the second term captures the word-of-mouth interaction effects. We next extend this monopoly model to mature markets with multiple brands.

3. Model Development

We first review the duopoly extension and then formulate an N -brand extension. Sorger (1989) extended the Sethi model to duopoly markets as follows:

$$\frac{dm_1}{dt} = \rho_1 u_1 \sqrt{1 - m_1} - \rho_2 u_2 \sqrt{m_1}, \quad m_1(0) = m_{10}, \quad (2)$$

$$\frac{dm_2}{dt} = \rho_2 u_2 \sqrt{1 - m_2} - \rho_1 u_1 \sqrt{m_2}, \quad m_2(0) = 1 - m_{10}, \quad (3)$$

where for each brand i , $i = 1, 2$, $m_i(t)$ represents market share, ρ_i denotes ad effectiveness, and u_i is the ad spending level of brand i that depends on both own and competitor's shares (i.e., $u_i(m_1, m_2)$). Both the spending levels and market shares are assumed to be strictly positive and positive fractions, respectively. In addition, by summing (2) and (3), we obtain $m_1(t) + m_2(t) = 1$ for all t , which represents the logical

consistency property. Although logical consistency is a desirable property for market share data, it may not hold for awareness levels or brand sales data. Hence, we also allow “market expansion” effects. Finally, we note that the exogenous decay constant δ in (1) is replaced in (2) and (3) by an endogenous loss due to the competitor’s advertising. However, in the context of awareness formation, competitor’s advertising might increase the awareness for own brands, for example, due to “confusion effects.” Consequently, Pauwels (2004, p. 604) urges researchers to permit both positive and negative effects of competitor’s advertising.

To incorporate market expansion effects, first we extend the Sorger (1989) formulation and allow the total awareness $M(t) = \sum_{i=1}^N A_i(t)$ to vary over time. Thus, for the N -brand oligopoly markets, we propose a generalized Sethi model

$$\frac{dA_i}{dt} = \underbrace{\rho_i u_i \sqrt{M - A_i}}_{\text{Gain due to own action}} - \sum_{\substack{j=1 \\ j \neq i}}^N \underbrace{\xi_{ij} u_j \sqrt{M - A_j}}_{\text{Loss or gain from competitor's action}}, \quad i \in I = \{1, 2, \dots, N\}, \quad (4)$$

where $u_i(\cdot)$ denotes the advertising effort based on prevailing awareness for all brands in the set $I = \{1, 2, \dots, N\}$. As before, the spending levels and awareness are assumed to be strictly positive. Second, to incorporate brand confusion effects, we capture potentially positive effects of competitor’s advertising (e.g., comparative advertisements) by permitting $\xi_{ij} < 0$ so that own awareness A_i can increase as the other brand’s ad spending u_j increases. Finally, if logical consistency is necessary (e.g., when using market share data), researchers can set $\xi_{ij} = \rho_j / (N - 1)$ and $M = 1$ to ensure $\sum_{i \in I} A_i(t) = 1$.

In sum, the proposed model not only allows competition across $N > 2$ brands, but also incorporates market expansion and brand confusion effects. We next assess the empirical validity of this model in describing brand awareness dynamics of competing brands.

4. Empirical Analyses

Below we present the awareness data, the estimation method, and the empirical results.

4.1. Continuous Tracking Data

To understand the awareness dynamics, we analyze field data for compact mid-priced cars from a continuous tracking study. Tracking studies offer the benefit that, even when brand managers do not have access to weekly sales data for competitors’ brands, they can commission market research companies (e.g., Millward Brown, Arbor Inc.) to conduct phone interviews and gather competitive awareness levels every

Table 1 Descriptive Statistics

Brands	Awareness levels		Gross rating points	
	Mean	Std. deviation	Mean	Std. deviation
Fiat Punto	8.10	3.53	171.74	235.54
Opel Corsa	1.86	0.79	65.07	140.31
Peugeot 206	1.62	1.03	58.91	135.40
Renault Clio	3.32	1.46	93.97	159.59
Ford Fiesta	1.46	0.97	74.41	117.16

week throughout the year (i.e., “continuous” tracking). Also, as Batra et al. (1995, p. 20) note, ad agencies prefer tracking data on “purer” intermediate measures (e.g., brand awareness, purchase intent) to assess advertising impact because nonadvertising factors such as salesforce effort or retail availability affect the final measures like brand sales or market shares.

In this tracking study, 350 phone interviews were conducted every day on a random sample of families in Italy. The information collected “... covers all players [i.e., brands] in the market. It covers the state of the play for that week in regard to people’s behavior, attitudes, brand awareness... This [awareness data] is then related to other information such as media data indicating what advertising was on during that week...” (see Sutherland 1993, Chap. 10, p. 112). Specifically, we use brand awareness of the five car brands—Fiat Punto, Opel Corsa, Peugeot 206, Renault Clio, and Ford Fiesta—in response to the question: “Which of these brands of cars have you seen advertised on television recently?” Besides the multibrand awareness time series, the other piece of information—the media spending patterns—consists of the weekly GRPs, which measures the exposure of advertisements to the target audience. GRPs are directly related to dollar expenditure and can be related to intermediate measures such as awareness (Rossiter and Percy 1997, p. 586). For example, Batra et al. (1995) fitted a logit model of awareness as a function of GRPs across 29 campaigns. Thus, our analysis of awareness-GRPs data comports with both the company’s information set and the extant practices (e.g., Sutherland 1993, Batra et al. 1995, Naik et al. 1998).

Thus, weekly time series of awareness and GRPs over 83 weeks for each of the five brands comprise the data set (see Luati and Tassinari 2005 for details). Table 1 presents the descriptive statistics. To analyze this time-series data, we next describe an approach for estimation, inference, and model selection of dynamic oligopoly models.

4.2. Extended Kalman Filter Estimation

In marketing, Xie et al. (1997) and Naik et al. (1998) pioneered the Kalman filter estimation of dynamic models. Specifically, Xie et al. (1997) studied nonlinear

but univariate dynamics of the Bass model, while Naik et al. (1998) estimated multivariate but linear dynamics of the modified Nerlove-Arrow model. However, the dynamic model in (4) constitutes a multivariate system of nonlinear differential equations. Hence, this paper marks the first nonlinear multivariate dynamic system estimation in marketing.

By discretizing Equation (4), we obtain a set of five equations for $i = 1, \dots, 5$ as follows:

$$A_{i,k+1} = A_{ik} + \rho_i u_{ik} \sqrt{M_k - A_{ik}} - \xi_{i2} u_{2k} \sqrt{M_k - A_{2k}} - \dots - \xi_{i5} u_{5k} \sqrt{M_k - A_{ik}}, \quad (5)$$

where A_i is a typical element of the five-dimensional state vector $\alpha_k = (A_{1k}, \dots, A_{5k})'$ and weeks $k = 1, \dots, T$. Equation (5) denotes a row of the simultaneous system of nonlinear difference equations with potential cross-equation coupling (i.e., it is not a typical regression model). The interequation coupling arises, for example, due to the presence of $M_k = \sum A_{i,k}$; the nonlinearity comes from the square-root formulation.

To estimate this nonlinear multivariate dynamic system, we apply the extended Kalman filter, which consists of four steps. (For several benefits of state space modeling, see the review chapter by Dekimpe et al. 2007.) First, to compactly represent the dynamic system in a state-space form, we let $\alpha_k = (A_{1k}, \dots, A_{5k})'$ denote the state vector, and $Y_k = (Y_{1k}, \dots, Y_{5k})'$ be the observed awareness levels. Then, the nonlinear multivariate state-space form is

$$Y_k = z\alpha_k + \varepsilon_k, \quad (6)$$

$$\alpha_k = \alpha_{k-1} + G(\alpha_{k-1}) + \nu_k,$$

where z is a 5×5 identity matrix, and $G(\cdot)$ is a vector-valued function of α_{k-1} . The error terms $\varepsilon_k \sim N(0, \Sigma_\varepsilon)$ and $\nu_k \sim N(0, \sigma_\nu^2 \mathbf{I})$, where \mathbf{I} denotes the 5×5 identity matrix. These errors can be interpreted as net effects of myriad factors not explicitly included in the model.

Second, the standard Kalman filter is not applicable because $G(\cdot)$ is a nonlinear function of state variables, and hence we apply the extended Kalman filter (EKF). The EKF recursively determines the mean and covariance of α_k , given the observed information history $H_{k-1} = \{Y_1, \dots, Y_{k-1}\}$ for each $k = 1, \dots, T$. We use the model-specific transition matrix $\mathbf{I} + \partial G / \partial \alpha'$ in the well-known EKF recursions (see Harvey 1994, p. 161) to obtain the sequence of means and covariances of α_k for $k = 1, \dots, T$.

Third, based on the means and covariances of α_k , we compute the log-likelihood of observing the awareness sequence $H_T = (Y_1, Y_2, \dots, Y_T)$, which is given by

$$L(\Theta; H_T) = \sum_{k=1}^T \text{Ln}(p(Y_k | H_{k-1})), \quad (7)$$

where $p(\cdot | \cdot)$ is the conditional density of Y_k given the information history up to the last period, H_{k-1} . The vector Θ contains the model parameters $(\rho_i, \xi_{ij})'$ and the error variances and the initial means of α_0 . We maximize (7) with respect to Θ to obtain the maximum-likelihood estimates

$$\hat{\theta} = \text{Arg Max } L(\Theta). \quad (8)$$

For correctly specified models, we obtain standard errors from the square root of the diagonal elements of the inverse of the matrix:

$$\hat{J} = \left[-\frac{\partial^2 L(\Theta)}{\partial \Theta \partial \Theta'} \right]_{\Theta=\hat{\theta}}, \quad (9)$$

where the Hessian of $L(\Theta)$ is evaluated at the estimated values $\hat{\theta}$. However, for misspecified models, we seek to make inferences robust to misspecification errors. To this end, we conduct Huber-White robust inferences (see White 1982) by computing the so-called sandwich estimator

$$\text{Var}(\hat{\theta}) = \hat{J}^{-1} \hat{V} \hat{J}^{-1}, \quad (10)$$

where V is a $K \times K$ matrix of the gradients of the log-likelihood function; that is, $V = G'G$, and G is $T \times K$ matrix obtained by stacking the $1 \times K$ vector of the gradient of the log-likelihood function for each of the T observations. In correctly specified models, $J = V$ and so both Equations (9) and (10) yield exactly the same standard errors (as they should); otherwise, we use the robust standard errors given by the square root of the diagonal elements of (10).

Finally, for model selection, we evaluate multiple information criteria (e.g., AIC, AIC_C, and BIC), which balance the trade-off between fidelity (enhance goodness-of-fit) and parsimony (employ few parameters). Specifically, we compute $\text{AIC} = -2L^* + 2K$, $\text{AIC}_C = -2L^* + T(T + K)/(T - K - 2)$, and $\text{BIC} = -2L^* + K \text{Ln}(T)$, where (L^*, T, K) denote the maximized log-likelihood, the sample size, and the number of parameters, respectively. The model to be retained is associated with the smallest values of the information criteria. By using multiple criteria, we gain convergent validity when these criteria suggest the same model as the best one, thus enhancing confidence. However, when multiple criteria indicate different models to be retained, researchers may apply the following guidelines: use AIC-type criteria to select models involving many parameters and small sample sizes (i.e., large K/T ratio) because they possess the “efficiency” property; use BIC-type criteria, which possess the consistency property, for settings with few parameters and large sample sizes (i.e., small K/T ratio). For further discussion of efficiency versus consistency, see McQuarie and Tsai (1998, p. 3) and Naik et al. (2007, Remark 2).

Table 2 Model Selection Using Multiple Information Criteria

Model	AIC _c	AIC	BIC
Complete model	637.79	353.64	418.95
Nested models ($\xi_{ij} = \rho_j/\lambda$)			
$\lambda = 1$	438.08	237.17	263.78
$\lambda = 2$	401.44	200.52	227.13
$\lambda = 3$	361.90	160.98	187.59
$\lambda = 4$	338.36	137.45	164.05
$\lambda = 5$	328.90	127.98	154.59
$\lambda = 6$	326.71	125.80	152.40
$\lambda = 7$	327.47	126.55	153.16
$\lambda = 8$	329.24	128.32	154.93
$\lambda = 9$	331.25	130.33	156.94
$\lambda = 10$	333.20	132.29	158.89
Full covariance model Σ_ϵ	378.08	132.35	183.15

4.3. Empirical Results

4.3.1. Model Selection and Parameter Estimation.

We apply the above approach to market data on five Italian car brands. We estimate the proposed model in (5), i.e., the complete model with five ρ_i parameters (one for each car brand), 20 ξ_{ij} parameters (four for each car brand), and $\Sigma_\epsilon = \sigma^2 \mathbf{I}_{5 \times 5}$. To determine the best model to retain, we compute multiple information criteria and display the results in Table 2. We find that this complete model is overparameterized. Specifically, it gets high scores on AIC, AIC_c, and BIC (see Table 2). Further inspection revealed that several of its estimated parameters were insignificant.

We then estimate 10 nested versions obtained by letting $\xi_{ij} = \rho_j/\lambda$, where $\lambda = 1, 2, \dots, 10$. Given the parsimonious specifications, we allow nonconstant variance across brands by relaxing the covariance matrix $\Sigma_\epsilon = \text{diag}(\sigma_1^2, \dots, \sigma_5^2)$. These nested versions have smaller scores on the information criteria (see Table 2). Particularly, the model with $\lambda = 6$ (shown in boldface) attains the minimum AIC, AIC_c, and BIC, so we retain this model with $\xi_{ij} = \rho_j/6$. More importantly, all three criteria recommend the same model to be retained, thus achieving convergent validity and enhancing confidence in this retained model.

If we want to estimate the full covariance matrix, we specify $\Sigma_\epsilon = PP'$, where P is a lower triangular matrix so that Σ_ϵ is positive definite (i.e., all its eigenvalues remain positive regardless of the magnitudes or signs of the elements in P in every quasi-Newton iteration of maximizing the log-likelihood function). However, our results show that most of the estimated elements in \hat{P} are not significant, resulting in the rejection of this specification based on the information criteria (see Table 2).

To further safeguard inference from other misspecification errors, we compute the robust standard errors via (10) for assessing the significance of estimated parameters. Table 3 presents the parameter estimates

Table 3 Extended Kalman Filter Estimates and Robust Inferences

Parameters of the retained model ($\lambda = 6$)	Filtered estimates	Robust <i>t</i> -ratios	Filtered estimates	Robust <i>t</i> -ratios
Brands	Ad effectiveness, ρ_i		Observation std. dev. σ_i	
Fiat Punto	0.4732	6.69	1.5063	5.70
Opel Corsa	0.4847	5.92	0.0909	4.88
Peugeot 206	0.5821	6.34	0.0004	0.60
Renault Clio	0.8138	6.62	0.3547	0.00
Ford Fiesta	0.5049	3.47	0.1415	3.15
Transition $\sigma_r \times 10^3$			0.2295	5.70
Max. log-likelihood			-51.90	

and robust *t*-ratios for the retained model. We find that all parameters have correct signs. Also, ad effectiveness of each of the five brands is statistically significant, indicating that the spending on television GRPs does affect a brand’s awareness. Furthermore, this impact of GRP on brand awareness differs by brand. Next, we consider how well this retained model predicts out-of-sample observations.

4.3.2. Out-of-Sample Forecast.

Using the retained model, we compute the one-step-ahead forecast of awareness levels via the cross-validation approach. Specifically, we re-estimate the model using the subsample of the first 52 observations and hold 31 observations to assess its predictive validity. Figure 1 displays the cross-validation results for the five brands (see the holdout period for actuals versus forecasts). These graphs show that the proposed model, fitted with subsample data, predicts the awareness levels in the holdout period satisfactorily. Table 4 quantifies the quality of forecast performance. For example, correlation between actual and forecasted awareness levels are high, ranging from 0.67 to 0.88 across five brands; similarly, forecast performance on other metrics (e.g., percentage observations within the confidence interval, mean squared error (MSE), mean absolute deviation (MAD)) are satisfactory. Thus, the proposed model not only fits the sample data, but also exhibits strong predictive performance.

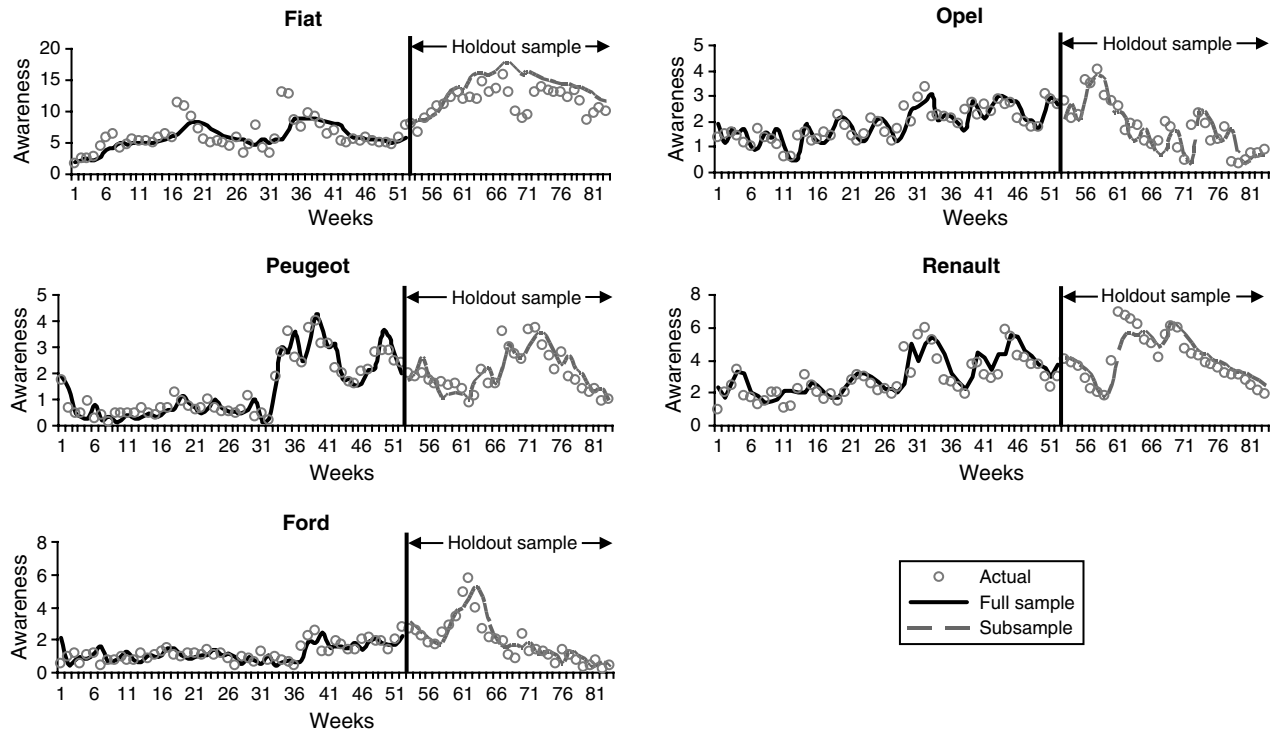
4.3.3. Model Comparison.

We close this section by comparing the retained model with an alternative model, namely, the *N*-brand Lanchester model whose specification is given by the following simultaneous system of coupled differential equations (see Fruchter 1999, Naik et al. 2005):

$$\begin{bmatrix} \dot{m}_1 \\ \vdots \\ \dot{m}_N \end{bmatrix} = \begin{bmatrix} f_1 \\ \vdots \\ f_N \end{bmatrix} - \begin{bmatrix} F & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & F \end{bmatrix} \begin{bmatrix} m_1 \\ \vdots \\ m_N \end{bmatrix}, \quad (11)$$

where $m_i = A_i/\sum A_i$ denotes brand *i*’s awareness share, $\dot{m}_i = dm_i/dt$, $f_i = \beta_i u_i$ is its marketing force, β_i

Figure 1 In-Sample Fit and Out-Of-Sample Forecasts for Car Brands



and u_i represent ad effectiveness and GRPs, respectively, and $F = \sum_{i=1}^N f_i$ is the total marketing force of all the brands.

To estimate the dynamic system (11), we apply the Kalman filter estimation described earlier to the estimation sample of 52 weeks. Next, based on this estimated model and average total awareness for the estimation sample, we predict one-step-ahead awareness levels of each of the five brands for the subsequent 31 weeks (i.e., the holdout sample, as before). Table 5 indicates the prediction results for the five-brand Lanchester model. Comparing the corresponding quantities in Tables 4 and 5, we observe that (i) the proposed model performs competitively on metrics such as correlation between actuals versus forecasts or percentage of observations within the 95% confidence interval; and (ii) the proposed model outperforms the alternative model on the formal metrics of MSE or MAD.

Table 4 Out-of-Sample Forecast Performance of the Generalized Sethi Model

Brands	Correlation between actuals and forecasts	Observations within the confidence interval (%)	MSE	MAD
Fiat Punto	0.67	80.6	9.1920	2.4129
Opel Corsa	0.83	90.3	0.3333	0.4469
Peugeot 206	0.81	93.5	0.2190	0.3656
Renault Clio	0.88	93.5	0.5186	0.5466
Ford Fiesta	0.88	80.6	0.4306	0.4828

Table 5 Out-of-Sample Forecast Performance of the Lanchester Model

Brands	Correlation between actuals and forecasts	Observations within the confidence interval (%)	MSE	MAD
Fiat Punto	0.56	64.5	14.0350	3.3072
Opel Corsa	0.84	96.8	1.1122	0.8844
Peugeot 206	0.70	96.8	1.0074	0.7957
Renault Clio	0.70	93.5	5.1102	1.8569
Ford Fiesta	0.85	96.8	1.5538	0.8717

Collectively, the empirical results furnish strong support, in terms of both fit and forecast, for the proposed N -brand model. Moreover, the estimation method—based on the extended Kalman filter for fitting nonlinear dynamic oligopoly models to market data—offers managers a systematic approach to estimate ad effectiveness and forecast awareness levels for their particular brands. Next, we derive the optimal strategies to pursue in a dynamic oligopoly.

5. Normative Analyses

We derive the optimal strategies and elucidate their substantive implications, relegating all the proofs to the online appendix (provided in the e-companion).¹

Starting from an initial awareness level A_{i0} , in response to advertising decisions to be made by all

¹ An electronic companion to this paper is available as part of the online version that can be found at <http://mansci.journal.informs.org/>.

other brands simultaneously, each brand manager decides the best course of action u_i^* to maximize the brand's total discounted present value (e.g., Fruchter and Kalish 1987, Sorger 1989),

$$\begin{aligned} V_i(A_{i0}) &= \int_0^\infty e^{-rt}(R(t) - c(u)) dt \\ &= \int_0^\infty e^{-rt}(m_i A_i(t) - u_i(t)^2) dt, \end{aligned} \quad i \in I = \{1, 2, \dots, N\}, \quad (12)$$

subject to the dynamic system (4). In (12), r denotes the discount rate, and $R(t)$ and $c(u)$ denote the revenues from and costs of building awareness over time, respectively. Specifically, m_i represents the revenues from an awareness point of brand i , and $c(u) = u^2$ is the convex cost function. Note that media companies (e.g., NBC, ABC) sell GRPs (denoted by u) and the advertiser pays for it in dollars. In other words, GRPs and dollar expenditure are two objects of the same exchange. As Tellis (2004, p. 45) notes, "GRPs are the most common measure for buying media time in the market today." Equation (12) specifies a quadratic relation between GRPs and advertising expenditures. Consequently, if we substitute $v = c(u) = u^2$ in the objective function and $u = \sqrt{v}$ in the dynamic state equations, a formulation consistent with previous studies (e.g., Batra et al. 1995, Fruchter 1999), we learn that the notion of diminishing returns (i.e., early dollar increments are more effective than the later ones) is incorporated in this formulation. Finally, $V_i(\cdot)$ captures the total discounted present value of brand i . To maximize (12), we solve the induced N -brand differential game and derive the closed-loop strategies in Nash equilibrium, which we state in the following proposition.

PROPOSITION 1. Maximizing (12) subject to (4), the optimal closed-loop advertising strategy of brand $i \in I$ is

$$u_i^* = \frac{\sqrt{M - A_i}}{2} \left((\rho_i + \xi_{ii}) \phi_i^i - \sum_{k=1}^N \xi_{ki} \phi_k^i \right), \quad (13)$$

and its value function

$$V_i = \sum_{j=1}^N \phi_j^i A_j, \quad (14)$$

where the coefficients ϕ_j^i are obtained from the following relations:

$$\begin{aligned} r \phi_i^i &= m_i + \sum_{j \in I, j \neq i} \frac{1}{2} \left((\rho_j + \xi_{jj}) \phi_j^j - \sum_{k \in I} \xi_{kj} \phi_k^j \right) \\ &\quad \cdot \left((\rho_j + \xi_{ji}) \phi_j^i - \sum_{k \in I} \xi_{ki} \phi_k^i \right), \end{aligned} \quad (15)$$

$$\begin{aligned} r \phi_j^i &= -\frac{1}{4} \left((\rho_i + \xi_{ii}) \phi_i^i - \sum_{k \in I} \xi_{ki} \phi_k^i \right)^2 \\ &\quad + \sum_{s \in I, s \neq j} \frac{1}{2} \left((\rho_s + \xi_{ss}) \phi_s^s - \sum_{k \in I} \xi_{ks} \phi_k^s \right) \\ &\quad \cdot \left((\rho_s + \xi_{ss}) \phi_s^i - \sum_{k \in I} \xi_{ks} \phi_k^i \right), \quad j \neq i. \end{aligned} \quad (16)$$

PROOF. See the online appendix.

It is remarkable that the dynamic model (4), which fits the market data well (see Table 3) and predicts the out-of-sample observations satisfactorily (see Figure 1), also leads to simple value functions, thus yielding closed-form optimal strategies. To extend this analysis to mature markets (i.e., when market expansion is negligible) or analyze market share data, we have to incorporate an additional constraint $\sum_{i \in I} A_i(t) = 1$ for every instant t . To ensure this logical consistency, we set $\xi_{ij} = \rho_j / (N-1)$ and $M = 1$ and obtain the following result:

PROPOSITION 2. In mature markets, a brand's optimal closed-loop advertising strategy is

$$u_i^* = \frac{\rho_i \sqrt{1 - A_i}}{2(N-1)} \left(N \phi_i^i - \sum_{j=1}^N \phi_j^i \right), \quad (17)$$

and its value function

$$V_i = \phi_0^i + \sum_{j=1}^N \phi_j^i A_j, \quad \forall i \in I, \quad (18)$$

where the coefficients (ϕ_0^i, ϕ_j^i) are obtained from the following $(N+1)$ relations:

$$r \left(\sum_{j=0}^N \phi_j^i \right) = m_i, \quad (19)$$

$$r \phi_i^i = m_i - \frac{\rho_i^2}{4(N-1)^2} \left(N \phi_i^i - \sum_{k \in I} \phi_k^i \right)^2, \quad (20)$$

$$\begin{aligned} r \phi_j^i &= -\frac{\rho_j^2}{2(N-1)^2} \left(N \phi_j^j - \sum_{k \in I} \phi_k^j \right) \left(N \phi_j^i - \sum_{k \in I} \phi_k^i \right), \\ &\quad \forall j \in I, j \neq i. \end{aligned} \quad (21)$$

PROOF. See the online appendix.

The optimal strategies in (17) reveal the *inverse allocation principle*: the greater the awareness level, the smaller the spending. A large (small) brand should spend proportionally less (more) than small (large) brands. Why? Because each brand spends proportional to the combined awareness of the remaining brands to compete optimally. This intuition comes from Equation (17), which shows that the optimal budget is proportional to $(u_i^*)^2 \propto 1 - A_i = \sum_{j \in I, j \neq i} A_j$.

We clarify that the inverse allocation principle differs from the competitive parity rule. Specifically, for two firms X and Y , competitive parity requires $u_X/A_X = u_Y/A_Y$, where (u_i, A_i) denote ad spends and awareness levels, respectively. Clearly, to maintain competitive parity, firms would spend proportional to their awareness levels (i.e., $u_i \propto A_i$), which is not only similar to the proportional-to-sales heuristic but, more importantly, opposite of the inverse allocation principle.

The substantive implication of this inverse allocation principle is that managers should build dominant brands because they would face less competitive resistance and afford to advertise more efficiently in the long run. From a life-cycle perspective, small up-and-coming brands should spend more on advertising to build awareness, whereas mature brands may “fly on automatic pilot” without advertising heavily and relying more on brand purchase and consumption experience to maintain awareness. Jones (1986, 1990) furnishes empirical evidence to corroborate this principle, noting that “...For large brands, the market share normally exceeds the advertising share; for smaller brands, the opposite is true” (Jones 1986, p. 100; emphasis in the original).

Interestingly, in mature oligopoly markets, equilibrium shares need not be of the form “us/(us+them),” as one expects in a duopoly market. To gain this counterintuitive insight, we derive the equilibrium share of awareness:

PROPOSITION 3. *For each brand in a mature market, the equilibrium share is given by*

$$\bar{A}_i = 1 - \frac{N-1}{B_i \times \sum_{j=1}^N B_j^{-1}} \quad \forall i \in I, \quad (22)$$

where $B_i = (\rho_i^2/2(N-1))(N\phi_i^i - \sum_{j=1}^N \phi_j^i)$.

PROOF. See the online appendix.

Now consider a duopoly market ($i = 1, 2$), where the equilibrium shares are $\bar{A}_i = B_i/(B_1 + B_2)$. In contrast, we realize that the equilibrium shares in a triopoly are $\bar{A}_i = 1 - 2B_1B_2B_3/B_i(B_1B_2 + B_2B_3 + B_1B_3)$ ($i = 1, 2, 3$), which differs from the expression “us divided by us plus them.” Hence, results from N -brand models can differ from those obtained using duopoly models, highlighting the importance of studying oligopoly generalizations. When the B_i s are equal across brands, each brand earns an equal share ($\bar{A}_i = 1/N$), as it should.

Finally, we investigate the effects of intensifying competition in mature markets. To gain insights, we simplify the analysis by assuming that brands are competing with “equals” and prove the result for the symmetric brands:

PROPOSITION 4. *In mature markets, the category ad spending increases as the number of brands increases.*

PROOF. See the online appendix.

Although this proposition seems intuitively obvious, the result is opposite of the extant findings in the literature. Specifically, Fershtman (1984) shows that category ad spending decreases as the number of brands increases; however, unlike our study, his market dynamics have not been empirically validated. As category ad spending increases as new brands enter, an important question arises: Is there a maximum number of brands that a mature product category can sustain?

To address this issue, we consider the total category value

$$\begin{aligned} \sum_{i \in I} V_i &= \int_0^\infty e^{-rt} \left(m - \sum_{i \in I} (u_i^*(t))^2 \right) dt \\ &= \frac{4m - \rho^2(\phi_1 - \phi_2)^2(N-1)}{4r}, \end{aligned} \quad (23)$$

where $(\phi_1 - \phi_2)$ is specified by Equation (D5) in the online appendix. We observe that the category value becomes negative as N increases, indicating that an upper bound for the number of brands exists. After algebraic manipulations, we obtain an upper bound (see the online appendix)

$$N^* = 3 + \frac{2r}{\sqrt{r^2 + m\rho^2} - r}, \quad (24)$$

which reveals that product categories will sustain three (or more) brands.

This finding—the smallest upper bound is three brands—closely relates to the so-called *Rule of Three*. At the individual level, consumers choose brands from a small consideration set, whose size ranges from two to eight brands across various product categories with a median of 3.0 for antacids, beers, deodorants, gasoline, over-the-counter medicines, pain relievers, and toothpastes (see Lilien et al. 1992, Table 2.11, p. 67). At the market level, Sheth and Sisodia (2002) suggest that only three major brands will eventually dominate any industry—for example, McDonald’s, Burger King, and Wendy’s in fast food; General Mills, Kellogg, and Post for breakfast cereals; and Nike, Adidas, and Reebok for sports shoes. Because different customers may be aware of different brands, these findings are appropriate for less heterogeneous markets. Our theoretical result that the smallest $N^* = 3$ not only affirms these empirical findings, but also reveals that managers can reduce the category size to three brands by increasing ad effectiveness, but not the media weight. That is, N^* decreases as ad effectiveness increases ($\partial N^*/\partial \rho < 0$), whereas it remains unchanged with an increase in media weight alone ($\partial N^*/\partial u = 0$). Thus, effective advertising can serve as a strategic device to reduce competition in mature product categories.

6. Conclusions

Awareness building in dynamic competitive markets is an important marketing activity. For example, Procter & Gamble bought the Gillette Company for \$57 billion when its book value was just \$11 billion in revenues and \$2 billion in earnings. Gillette's intangible property—not reflected in its accounting books—is the awareness in consumers' minds for brand names like Sensor, Mach 3, Duracell, Oral B, and Braun. Although existing marketing models investigate how to build brand awareness, these models ignore the presence of competition (see Mahajan et al. 1984). This gap between theory and practice motivates us to extend awareness models to mature markets by explicitly incorporating oligopolistic competition. To bridge this gap, we propose a dynamic oligopoly model (§3), develop an estimation approach (§4.2), validate the proposed model empirically (§4.3), and analyze the implied differential game (§5). More importantly, managers can apply the estimation approach to awareness data from their particular product markets to assess ad effectiveness and predict awareness levels for their own and competitors' brands.

We conclude by identifying five avenues for future research. The first avenue is to extend model (4) to multiple media to incorporate the effects of cross-media interactions or synergies (e.g., Naik and Raman 2003). The second is to incorporate cross-competitor interactions by specifying ξ_{ij} in model (4) to functionally depend on u_i so that we can test whether a brand's own advertising dampens the awareness loss to competitors. Third, to compute optimal strategies and value functions empirically, managers should estimate the value of \hat{m}_i for each brand. To this end, they can use their private information to construct a dependent variable as "price minus variable cost multiplied by weekly sales units" and regress it on prevailing awareness levels across weeks. Fourth, we encourage further research on the Rule of Three, the idea that category ad spending increases with an additional entrant raises the question of how markets themselves evolve, entailing a comparison of a large market and a small market at various points in time. Finally, future researchers may derive normative implications when the shape of the response function is not globally concave (Vakratsas et al. 2004). We believe these efforts would improve the theory and practice of marketing communications.

7. Electronic Companion

An electronic companion to this paper is available as part of the online version that can be found at <http://mansci.journal.informs.org/>.

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