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Optimal Advertising When Envisioning a Product-Harm Crisis

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How should forward-looking managers plan advertising if they envision a product-harm crisis in the future? To address this question, we propose a dynamic model of brand advertising in which, at each instant, a nonzero probability exists for the occurrence of a crisis event that damages the brand's baseline sales and may enhance or erode marketing effectiveness when the crisis occurs. Because managers do not know when the crisis will occur, its random time of occurrence induces a stochastic control problem, which we solve analytically in closed form. More importantly, the envisioning of a possible crisis alters managers' rate of time preference: anticipation enhances impatience. That is, forward-looking managers discount the present—even when the crisis has not occurred—more than they would in the absence of crisis. Building on this insight, we then derive the optimal feedback advertising strategies and assess the effects of crisis likelihood and damage rate. We discover the crossover interaction: the optimal precrisis advertising decreases, but the postcrisis advertising increases as the crisis likelihood (or damage rate) increases. In addition, we develop a new continuous-time estimation method to simultaneously estimate sales dynamics and feedback strategies using discrete-time data. Applying the method to market data from the Ford Explorer's rollover recall, we furnish evidence to support the proposed model. We detect compensatory effects in parametric shift: ad effectiveness increases, but carryover effect decreases (or vice versa). We also characterize the crisis occurrence distribution that shows that Ford Explorer should anticipate a crisis in 2.1 years and within 6.3 years at the 95% confidence level. Finally, we find a remarkable correspondence between the observed and optimal advertising decisions.

Key words: product-harm crisis; optimal advertising; stochastic optimal control; random stopping time problem; Kalman filter; Ford Explorer rollover
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1. Introduction

Companies face a realistic chance of encountering product-harm crisis (e.g., Toyota vehicles' unintended acceleration incidents, British Petroleum (BP)'s oil spill, the Ford-Firestone tire recall). Because of the crisis of unintended acceleration in the winter of 2010, Toyota recalled 8 million cars, lost 16% sales, and dropped $12 billion in market value. BP's oil spill could have resulted in a bankruptcy. The Ford-Firestone crisis resulted in claims worth $3.5 billion and ended their century-old relationship. Thus, product-harm crisis arrives unannounced and damages a brand's sales and profitability.

To help managers respond to crisis events, consulting companies offer advice such as telling the truth to relevant groups, containing the crisis, repairing the damage, and rebuilding the brand (Augustine 1995, O'Donnell 2009). To understand customers' response, experimental studies investigate the effects of hypothetical crises on consumer expectations (Dawar and Pillutla 2000), commitment to the brand (Ahluwalia et al. 2000), brand loyalty (Stockmyer 1996), perceived locus of the problem (Griffin et al. 1991), or prior corporate social responsibility (Klein and Dawar 2004). To gauge the extent of damage to baseline sales and marketing effectiveness, empirical studies estimate dynamic sales models after the crisis occurs. For example, the study by Van Heerde et al. (2007) shows that a peanut butter crisis in Australia damaged the brand's baseline sales and attenuated the effectiveness of its marketing mix activities. Thus, the literature studies ex post consequences of crises using the three types of studies: managerial, experimental, and empirical. But managers should expect a crisis to occur unexpectedly, so how should they make ex ante advertising decisions differently if they envision a crisis in the future?

The extant literature, to the best of our knowledge, contains no normative study that sheds light on this question. Consequently, we do not know the effects of incorporating the risk of crisis in advertising decisions. Specifically, should forward-looking managers increase—or decrease—precrisis advertising if crisis likelihood increases? If managers anticipate that brand sales will plummet, how should they optimally set an advertising budget ex ante (i.e., before the crisis)? How would they estimate the crisis likelihood using sales advertising data? How much should they spend if they aspire to make an impressive comeback (e.g., Tylenol contamination)? Should competing brands exploit the misfortunes of the focal brand?

These questions are simple to ask but hard to answer. To see this point, consider the decision dilemma: should managers increase or decrease the precrisis advertising if they envision a crisis in the future? One argument suggests that precrisis advertising would insulate the brand from negative publicity when it occurs, which requires managers to increase ad spending (Cleeren et al. 2008). On the other hand, investments in precrisis advertising would be partly wasted because of the likely damage to baseline sales, which requires them to decrease ad spending. Both arguments—the insulation effect and investment effect—are valid, and so we cannot resolve the opposing recommendations without formal analysis.

Hence, we formulate a dynamic advertising model in which, at each instant, a nonzero probability exists for the occurrence of a crisis event that hurts the brand's sales and may erode or enhance marketing effectiveness when the crisis occurs. The random occurrence time induces a stochastic control problem, which we solve analytically to derive the optimal feedback advertising strategies in closed form. Applying comparative statics analysis, we deduce the effects of crisis likelihood and damage rate. We next extend the basic model to incorporate recovery dynamics and competition. In addition, we investigate the role of size-dependent insulation effect (i.e., larger brands are less vulnerable to sales loss), investments in quality (i.e., improving quality to reduce crisis likelihood), and optimal pricing strategy. Finally, we apply the basic model to Ford Explorer's rollover crisis, which damaged the baseline sales by 35% immediately after the crisis (compared with 16% for Toyota's recent crisis).

The above theoretical and empirical analyses make three broad contributions. First, the analytical model introduces a novel feature that the crisis event occurs at a random time in the future. Previous dynamic models incorporated uncertainty via continuous Brownian motion (i.e., the Wiener process), which represents the effects of several small shocks whose net impact, on average, is zero. In contrast, crisis causes uncertainty via a discrete event whose net impact is not zero. We furnish a general framework to study uncertainty from rare catastrophic events, where the events are large and discrete rather than small and continuous. The resulting dynamic model augments the class of control-theoretic models in marketing (e.g., see Feichtinger et al. 1994, Sethi and Thompson 2000) by introducing a random stopping problem (see §3), which facilitates the study of long-term optimal decisions in the presence of rare but catastrophic events (Boukas et al. 1990, Haurie and Moresino 2006).

Second, besides this modeling contribution, we offer seven new propositions on managing advertising in the presence of a potential product-harm crisis. Specifically, the mere act of envisioning a crisis alters managers' time preference: crisis likelihood enhances impatience. That is, forward-looking managers discount the present—even in the precrisis regime, when the crisis has not occurred—more than they would in the absence of crisis. In resolving the dilemma described earlier, we discover the *crossover* interaction: managers should decrease the precrisis advertising and increase the postcrisis advertising as crisis likelihood (or damage rate) increases. Conserve spending now; defend sales later. We elucidate the intuition in §4.

Third, this paper is the first to develop a method for simultaneously estimating both sales dynamics and feedback strategies in continuous-time using discrete-time data (see §5.2). Empirical results establish the presence of compensatory effects in parametric shift: ad effectiveness increases, but carryover effect decreases (or vice versa) after the crisis. This finding complements and augments the previous results on uniform attenuation reported by Van Heerde et al. (2007). Moreover, we show how to estimate the crisis likelihood and characterize the distribution of random crisis occurrence. Results suggest that Ford Explorer should expect a crisis in about two years and within six years at the 95% confidence level. Finally, by comparing the observed and optimal ad spending, we find a remarkable correspondence after the crisis: actual advertising is 1.7% above the optimal expenditures for the Ford Explorer, 4.4% for the Jeep Cherokee, and 1.9% for the Toyota 4Runner. Thus, the proposed model comports with the managers' actions.

The rest of this paper proceeds as follows. Section 2 develops the dynamic brand sales model with a random jump resulting from the crisis event. Section 3 evaluates the long-term profit impact of the crisis event, and §4 derives the optimal feedback advertising strategies and the effects of crisis likelihood and sales damage. Section 5 presents the market data from Ford Explorer rollover recall and lends empirical support for the proposed model. Section 6 discusses several related issues (e.g., size-dependent insulation effect, investments in quality, pricing decisions), and §7 concludes by identifying avenues for future research.

2. Model Development

2.1. Brand Sales Dynamics

Let $u(t)$ denote a firm's advertising spending at time t to build the brand's sales $S(t)$. For example, Ford Explorer's advertising $u(t)$ exceeds \$1 million every week (see Table 1 later). The resulting growth in brand sales can be described by the nonlinear extension of the Vidale-Wolfe model formulated by Sethi (1983):

$$\frac{dS}{dt} = \beta_j \sqrt{u_j(t)}\sqrt{M(t) - S(t)} - \delta_j S(t), \tag{1}$$

where β_j represents ad effectiveness, $\sqrt{u_j(t)}$ captures the diminishing returns to advertising, $(M - S)$ is the untapped market with M as the market size, δ_j is the decay rate, and the index $j = 1$ and 2 denotes the pre- and postcrisis regimes, respectively.

Equation (1) states that the ad spending $u(t)$ results in sales growth dS during the time interval dt by capturing some portion of the untapped market; in the absence of advertising, sales decline at the rate proportional to the prevailing sales level. Previous advertising studies have used this model in empirical and analytical research (e.g., Feichtinger et al. 1994, Erickson 2003, Naik et al. 2008, Erickson 2009b).

The main difference from previous models is the probabilistic switch in the regimes as a result of a crisis. Equation (1) specifies the evolution of the one continuous state variable $S(t)$ and one discrete state variable ($j = 1$ or $j = 2$) that identifies the regime in which the brand operates (i.e., pre- or postcrisis). The index j indicates that all model parameters differ before and after the crisis. In the study on crisis impact, Van Heerde et al. (2007) find that ad effectiveness and carryover effect drop below their corresponding precrisis levels. Although Van Heerde et al. (2007) assume that they know when the crisis occurred in their data (i.e., deterministic regime switch), the crisis occurs at an unknown date ex ante when planning for the next year's advertising budget. To this end, we describe the probabilistic evolution of this binary state variable.

2.2. Probabilistic Regime Switch as a Result of a Crisis

Because crisis strikes at an unknown instant in the future, the timing of crisis occurrence is random. Let $\{\Gamma(t)\colon t \geq 0\}$ denote the crisis occurrence process in which the crisis may occur at any instant t with probability $\chi \in (0, 1)$. Furthermore, let T be the random date at which the crisis actually occurs. Then, the horizon splits into two subintervals: the precrisis regime ($j = 1$), where $t \in [0, T)$ before the crisis; and the postcrisis regime ($j = 2$), where $t \in [T, \infty)$ after the crisis. Consequently, the process $\{\Gamma(t)\colon t \geq 0\}$ is a jump process with the jump rate defined by

$$\lim_{dt \to 0} \frac{P[\Gamma(t+dt) = 2 \mid \Gamma(t) = 1)]}{dt} = \chi \quad \text{and}$$
$$\lim_{dt \to 0} \frac{P[\Gamma(t+dt) = 1 \mid \Gamma(t) = 2)]}{dt} = 0. \tag{2}$$

To complete the model formulation, we specify the initial conditions for the pre- and postcrisis regimes. At $t = 0$, the initial sales level is

$$S(0) = S_{10}. \tag{3}$$

Based on Van Heerde et al. (2007), the baseline sales drop after the product-harm crisis, and we capture this "damage" as follows:

$$S(T^+) = (1 - \phi) S(T^-), \tag{4}$$

where $S(T^+)$ and $S(T^-)$ represent the sales levels just after and right before the crisis event, respectively. The fraction ϕ is the damage rate: the larger the damage rate, the sharper the drop in the baseline sales, which hurts the long-term profit.

3. Long-Term Profit Impact of a Crisis

Here, we formulate the advertising decision problem, evaluate the long-term profit as a function of crisis likelihood, and discover how crisis alters decision making.

3.1. Advertising Decisions in the Presence of a Crisis

In a continuous-time framework, the net present value of brand advertising is given by $J(u) = \int_0^\infty e^{-\rho t} \pi(S(t), u(t))\, dt$, where ρ denotes the discount rate, $\pi(S, u) = mS - u$ is the profit function, and m is the unit margin (e.g., see the review articles by Sethi 1977, Little 1979, Feichtinger et al. 1994). When brand managers envision a crisis event at the random date T in the future, they compute the profits from the pre- and postcrisis regimes. Let (m_1, m_2) denote the margins in the two regimes, respectively. Although the brand's margin can be the same in both the regimes, we relax them in the following analysis. For example, $m_2 < m_1$ can be thought of as an increase in the variable cost to improve quality (e.g., Tylenol designed tamperproof packaging). Then the net present value from each regime is

$$J_1(u_1) = \int_0^T e^{-\rho t} \pi(S(t), u_1(t))\, dt \quad \text{and} \tag{5a}$$

$$J_2(u_2) = \int_T^\infty e^{-\rho t} \pi(S(t), u_2(t))\, dt. \tag{5b}$$

In Equation (5a), J_1 is the total discounted profit from $t = [0, T)$, and it accrues at time $t = 0$. Likewise, Equation (5b) yields the total discounted profit from $t = [T, \infty)$, and the resulting J_2 becomes available at time $t = T$.

Note that T is a random variable because the manager does not know ex ante at $t = 0$ the exact date when the crisis will occur. The random date T takes any value on the time line $[0, \infty)$. Consequently, the net present value J_1 is also a random variable; J_1 is small if T occurs early and large if T occurs later. Hence, by taking the expectation over all the possible values of T, we compute the expected net present value of profit from both the regimes:

$$J(u_1, u_2) = E[J_1(u_1) + e^{-\rho T} J_2(u_2)], \tag{6}$$

where the expectation $E[\cdot]$ is taken with respect to the crisis occurrence process in (2). The discount factor $e^{-\rho T}$ appears in (6) because J_1 accrues at $t = 0$ and J_2 at $t = T$, requiring the discounting of J_2 back to $t = 0$ to add the two long-term profits.

The resulting expected long-term profit in (6) depends on the choice of advertising feedback strategies $(u_1(S), u_2(S))$. The forward-looking managers take into account the intertemporal trade-off in spending resources now versus later. In addition, they incorporate the probability χ with which a crisis might strike. We denote the optimal advertising strategies (u_1^*, u_2^*), which are feedback solutions to the optimization problem:

$$(u_1^*(S), u_2^*(S)) = \arg\max J(u_1(S(t)), u_2(S(t))), \tag{7}$$

subject to the sales dynamics in (1), the random crisis occurrence process in (2), and the initial values in pre- and postcrisis periods via Equations (3) and (4), respectively.

Equation (7) represents a stochastic control problem because it involves a discrete random event of crisis occurrence (because of Equation (2)). Previous marketing studies formulated stochastic control problems using the Wiener process (e.g., Rao 1986, Nguyen 1997, Raman and Chatterjee 1995, Raman and Naik 2004, Prasad and Sethi 2004). The uncertainty structure of the Wiener process represents many small shocks of uncertain magnitude at every instant whose net impact on average is zero. However, the product-harm crisis induces a different uncertainty structure qualitatively. Specifically, it involves timing uncertainty, damages the baseline sales, and attenuates or amplifies marketing effectiveness. In other words, we face a large shock at an uncertain time whose net impact on average is not zero. To solve this discrete event stochastic control problem, we evaluate the expectation in (6).

3.2. Random Stopping Problem

When the crisis occurs, the precrisis regime ends. The timing of its occurrence is random. Let $f(t)$ and $F(t)$ denote the probability density and cumulative distribution functions of the crisis occurrence process in (2), respectively. Because the crisis can occur at any instant t, given that it has not occurred thus far in the interval $[0, t)$, the crisis hazard rate $h(t) = \chi$. Then the distribution function is given by $F(t) = 1 - \exp(-\int_0^t h(s)\,ds)$, and the density function is given by $f(t) = h(t) \times (1 - F(t)) = \chi \exp(-\int_0^t \chi\,ds)$. Using this density function $f(t)$, we evaluate the expectation in (6) as follows:

$$\begin{aligned} J(u_1, u_2) &= E\left[\int_0^t e^{-\rho s}\pi(s)\,ds + e^{-\rho t} J_2\right] \\ &= \int_0^\infty \left[\int_0^t e^{-\rho s}\pi(s)\,ds + e^{-\rho t} J_2\right](\chi e^{-\chi t})\,dt \\ &= I_1 + \chi \int_0^\infty e^{-(\rho+\chi)t} J_2\,dt. \end{aligned} \tag{8}$$

In the appendix, we prove that

$$I_1 = \int_0^\infty e^{-(\rho+\chi)t}\pi(t)\,dt. \tag{9}$$

Finally, substituting (9) into (8) and simplifying, we obtain the total long-term profit from both regimes:

$$J(u_1, u_2, \chi) = \int_0^\infty e^{-(\rho+\chi)t}\{\pi(S(t), u_1(t)) + \chi J_2(u_2(t))\}\,dt. \tag{10}$$

Equation (10) is the long-term profit as a function of advertising decisions and the crisis likelihood. It reveals an insight into how the presence of a crisis event alters decision making.

To see this insight, compare Equations (5) and (10) and observe that the discount rate increases from ρ to $(\rho + \chi)$. Thus, the envisioning of a crisis alters the manager's rate of time preference. Specifically, forward-looking managers discount the present—even in the precrisis regime, when the crisis has not occurred—more than they would in the absence of crisis. Thus, they become more impatient as they envision crisis scenarios, leading us to the following.

Proposition 1. *Crisis likelihood enhances impatience.*

4. Effects of Crisis on Optimal Advertising

We first formulate and solve the value functions to derive the optimal feedback advertising strategies and then deduce the effects of crisis likelihood and damage rate.

4.1. Value Functions in Pre- and Postcrisis Regimes

The goal of forward-looking managers is to maximize the long-term profit by selecting the best ad spending at each instant t in both the pre- and postcrisis regimes. Formally stated, they should consider all possible strategies $(u_1(S), u_2(S))$ and select the best ones $(u_1^*(S), u_2^*(S))$ that maximize the long-term profit Equation (10). When they use the optimal advertising strategies, the largest value of the long-term profit in (10) is given by

$$\begin{aligned} V(S_{10}, S(T^+)) &= \max_{u_1, u_2} J(u_1(t), u_2(t), \chi) \\ &= \max_{u_1, u_2} \int_0^\infty e^{-(\rho+\chi)t}\{m_1 S(t) - u_1 + \chi J_2(u_2)\}\, dt, \end{aligned} \tag{11}$$

starting from any initial sales S_{10} before the crisis and initial sales level $S(T^+)$ just after the crisis. In Equation (11), forward-looking managers incorporate the profit from the second regime, $J_2(\cdot)$, when deciding their actions in the precrisis regime. Consequently, the maximum value of $J_2(\cdot)$ is necessary to maximize (11). Hence, we apply the backward induction principle and start with the second regime first.

In the second regime, let $W(S(T^+)) = \max_{u_2} J_2(u_2(t))$ represent the value function, which yields the maximum long-term profit, starting from the new baseline sales $S(T^+)$ after the crisis. Because the time left after the crisis from $[T, \infty)$ is the same as $[0, \infty)$, we translate the origin and solve the control problem:

$$W(S(T^+)) = \max_{u_2} \int_0^\infty e^{-\rho t}\{m_2 S(t) - u_2\}\, dt$$

subject to

$$\begin{aligned} &\frac{dS}{dt} = \beta_2\sqrt{u_2(t)}\sqrt{M(t) - S(t)} - \delta_2 S(t), \\ &S(T^+) = (1-\phi)S(T^-). \end{aligned} \tag{12}$$

In the appendix, we solve for the value function and derive the optimal postcrisis feedback advertising strategy.

Substituting the resulting value function $W(S(T^+))$ from (12) in (11), we then solve the control problem in the first regime, which is given by

$$V(S_{10}) = \max_{u_1} \int_0^\infty e^{-(\rho+\chi)t}\{m_1 S(t) - u_1 + \chi W((1-\phi)S(t))\}\, dt \tag{13}$$

subject to

$$\frac{dS}{dt} = \beta_1\sqrt{u_1(t)}\sqrt{M(t) - S(t)} - \delta_1 S(t), \quad S(0) = S_{10}.$$

In the appendix, we also derive the optimal precrisis feedback advertising strategy. Next, we state both the optimal strategies in Proposition 2.

4.2. Optimal Advertising

Let λ_1 and λ_2 denote the long-term profitability in the pre- and postcrisis regimes, respectively. We interpret λ_j as the marginal long-term profit as a result of incremental sales in regime j (e.g., Kamien and Schwarz 2003, p. 136). In the appendix, we solve the Hamilton–Jacobi–Bellman equation associated with Equations (12) and (13) to derive the optimal feedback advertising strategies, which we present in the following.

Proposition 2 (Optimal Advertising Strategies). *The optimal pre- and postcrisis feedback advertising strategies, respectively, are given by*

$$u_1^*(S(t)) = (M(t) - S(t))(0.5\beta_1\lambda_1)^2 \quad \text{and}$$
$$u_2^*(S(t)) = (M(t) - S(t))(0.5\beta_2\lambda_2)^2,$$

where

$$\lambda_1 = \frac{-2(\rho+\delta_1+\chi) + 2\sqrt{(\rho+\delta_1+\chi)^2 + (m_1 + \chi(1-\phi)\lambda_2)\beta_1^2}}{\beta_1^2}$$

and

$$\lambda_2 = \frac{-2(\rho+\delta_2) + 2\sqrt{(\rho+\delta_2)^2 + m_2\beta_2^2}}{\beta_2^2}.$$

Proof. See the appendix.

The above feedback advertising possesses the desirable property of *subgame perfectness*. That is, the optimal advertising spending is optimal not only when sales evolve along the optimal state trajectory of the starting game but also when sales depart from the optimal trajectory in *any* subgame, which may (i) occur probabilistically at any time because of a crisis, (ii) induce any possible sales drop, and (iii) alter any model parameter(s). This property is also known as Markov perfectness (Zaccour 2008) or strong time consistency (Başar and Olsder 1999).

In the absence of crisis ($\chi = 0$), optimal ad spending $u_1^*(S) = u_2^*(S)$ (as it should be), and it depends on the ad effectiveness β, the margin m, and the decay and discount rates (δ, ρ). This result informs managers to spend more on advertising when ad effectiveness or brand profitability increases (for example, because of a new television commercial that is more effective or a new technology that reduces variable costs). Furthermore, managers should spend more when sales are low and less when sales are high—that is, spend intensively when the untapped market is large.

In the presence of a potential crisis, the forward-looking managers should alter their pre- and postcrisis ad spending based on the following two factors: the likelihood χ of crisis occurrence (which enhances impatience) and the damage rate ϕ during the crisis (which leads to the loss in baseline sales). To gain further insights and contribute new results to marketing science, we conduct comparative statics analysis of optimal strategies with respect to the above factors.

4.3. Crisis Likelihood Effects

Before we present the results, we restate the decision dilemma: should managers increase or decrease advertising as crisis likelihood increases? One argument—the insulation effect—favors an increase in precrisis advertising because advertising builds brand equity that "buffers" the potential damage after the crisis (Cleeren et al. 2008, p. 262). A counterargument—the investment effect—recommends a decrease in precrisis advertising to offset the loss in profit they expect as a result of the crisis (e.g., lost baseline sales).

Both arguments are valid. Indeed, the long-term profit embodies the insulation and investment effects of crisis. Specifically, in Equation (10), the crisis likelihood χ appears in both the exponent term and the multiplier of the postcrisis profit J_2. The exponent term, as stated in Proposition 1, enhances managers' impatience, which reduces the level of effort they exert in the precrisis regime and thus decreases the precrisis ad spending as χ increases. In contrast, the multiplier term increases the total profit, thereby increasing the precrisis ad spending as χ increases. Because the two opposing forces coexist, the extant literature cannot shed light in general (i.e., for every feasible parameter value) on whether optimal advertising should increase or decrease as χ increases. Hence, to resolve this dilemma, we apply comparative statics analysis to obtain the new findings.

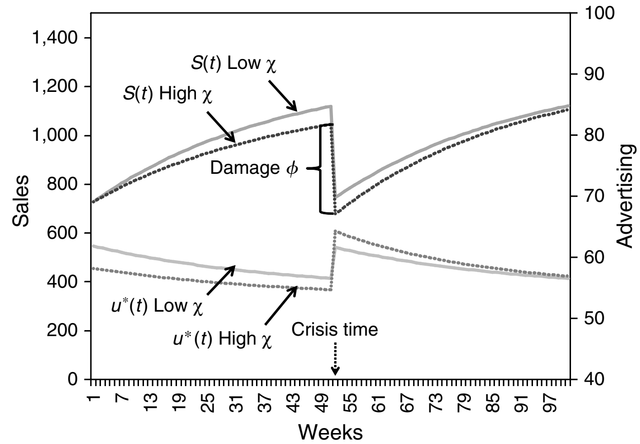
Proposition 3 (Crisis Likelihood Effects). *As the crisis likelihood χ increases, the optimal precrisis advertising decreases, whereas the optimal postcrisis advertising increases.*

Proof. See the appendix.

Why should managers reduce advertising when considering the mere possibility of a crisis? Because managers become impatient as χ increases, as per Proposition 1, so they reduce the brand-building effort. It is remarkable that even though crisis affects the postcrisis regime, it influences precrisis decisions. Why? The answer is that the principle of backward induction requires managers to look forward and then reason backward to make dynamically optimal decisions.

Figure 1 illustrates the trajectories of sales and advertising in the presence of low and high likelihoods of crisis. First, it shows that precrisis ad spending is lower for higher χ at every $t < T$. Second, at the crisis time T, advertising increases to recover the lost sales. Third, we observe a larger postcrisis advertising for higher χ at every $t > T$.[1] This result follows from

Figure 1 **Crisis Likelihood Effects on Advertising and Sales Trajectories**



the mediating effects of χ via $u_1^* \to S_1(t) \to S_1(T^-) \to S_2(T^+) \to S_2(t) \to u_2^*$, reported in Figure 2. The direct effect of χ (or ϕ) on u_2^* is zero.

Thus we discover the *crossover* interaction: u_1^* decreases but u_2^* increases when managers envision a crisis. They should conserve resources now to recover sales later.

Finally, we emphasize that constant hazard rate provides a benchmark to learn how managers should adapt their advertising in the presence of negative or positive duration dependence. Specifically, Proposition 3 shows that optimal advertising should be reduced if the hazard rate increases (positive dependence) and enhanced if it decreases (negative dependence). Explicit time-varying hazard analysis (not reported here) for negative and positive duration dependence comport with these predictions.

4.4. Damage Effects

Proposition 4 (Damage Effects). *As the damage rate ϕ increases, the optimal precrisis advertising decreases, whereas the optimal postcrisis advertising increases.*

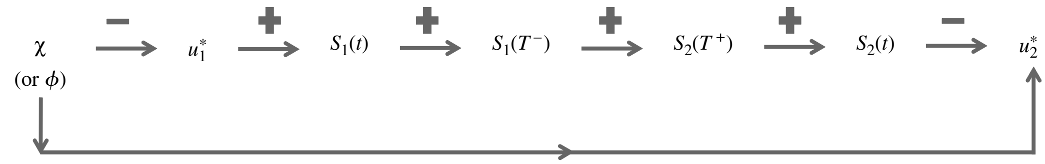
Proof. See the appendix.

The intuition for this result is as follows. A crisis damages the baseline sales. When managers anticipate this damage, they recognize the potential loss in profit in the postcrisis regime. Because they maximize the sum of profits from both regimes, a reduction in precrisis ad spending offsets the anticipated loss in the postcrisis regime. In other words, they plan ahead by reducing the precrisis ad spending to compensate future losses. Hence, precrisis ad spending decreases as the damage rate increases.

The postcrisis sales are affected by the combination of two adverse forces. First, the reduction in precrisis advertising suppresses the sales trajectory, leading to the lower sales $S_1(T^-)$. Second, the damage rate ϕ further reduces $S_2(T^+)$ via Equation (4). Hence, postcrisis advertising increases to compensate for the double

[1] We acknowledge the contributions of the associate editor in this mediation analysis.

Figure 2 Mediation Effects



whammy: brand sales starts lower at $t = T$ and drops even further because of ϕ. See Figure 1.

4.5. Recovery

How would advertising strategies differ when a brand manager envisions recovery after the crisis? To investigate this question, let the crisis last for a random duration so that the brand recovers at time τ. To account for this "comeback" to the precrisis state, we augment the process $\Gamma(t)$ as follows:

$$\lim_{dt\to 0} \frac{P[\Gamma(t+dt)=2 \mid \Gamma(t)=1)]}{dt} dt = \chi \quad \text{and} \qquad \lim_{dt\to 0} \frac{P[\Gamma(t+dt)=1 \mid \Gamma(t)=2)]}{dt} dt = \omega, \tag{14}$$

where $\omega \in (0, 1)$ is the intensity of the recovery process. The recovery is characterized as normal ($\theta = 0$) or impressive ($\theta > 0$), which bumps up the baseline sales as follows:

$$S(\tau^+) = (1+\theta)S(\tau^-). \tag{15}$$

To find the optimal strategy u_2 in the crisis regime, the brand manager faces the dynamic optimization problem:

$$W(S(T^+)) = \max_{u_2} \int_0^\infty e^{-(\rho+\omega)t}\{m_2 S(t) - u_2 + \omega V((1+\theta)S(t))\}\, dt$$

$$\text{subject to} \tag{16}$$

$$\frac{dS}{dt} = \beta_2 \sqrt{u_2(t)}\sqrt{M(t)-S(t)} - \delta_2 S(t), \quad \text{and}$$

$$S(T^+) = (1-\phi)S(T^-).$$

In the precrisis regime, to obtain u_1, the brand manager faces the dynamic optimization problem:

$$V(S(\tau^+)) = \max_{u_1} \int_0^\infty e^{-(\rho+\chi)t}\{m_1 S(t) - u_1 + \chi W((1-\phi)S(t))\}\, dt$$

$$\text{subject to} \tag{17}$$

$$\frac{dS}{dt} = \beta_1 \sqrt{u_1(t)}\sqrt{M(t)-S(t)} - \delta_1 S(t), \quad \text{and}$$

$$S(\tau^+) = (1+\theta)S(\tau^-).$$

We solve the optimization problems in (16) and (17) and present the results in the next proposition.

Proposition 5 (Recovery). *In the presence of recovery dynamics, the optimal feedback advertising strategies are given by*

$$u_1^*(S(t)) = (M(t) - S(t))(0.5\beta_1\lambda_1)^2 \quad \textit{and}$$

$$u_2^*(S(t)) = (M(t) - S(t))(0.5\beta_2\lambda_2)^2,$$

where λ_1 and λ_2 are the solutions to the algebraic equations:

$$\lambda_1 = \frac{-2(\rho+\delta_1+\chi) + 2\sqrt{(\rho+\delta_1+\chi)^2 + (m_1 + \chi(1-\phi)\lambda_2)\beta_1^2}}{\beta_1^2},$$

and

$$\lambda_2 = \frac{-2(\rho+\delta_2+\omega) + 2\sqrt{(\rho+\delta_2+\omega)^2 + (m_2 + \omega(1+\theta)\lambda_1)\beta_2^2}}{\beta_2^2}.$$

Proof. See the appendix.

Because λ_2 is an increasing function of θ, managers should increase advertising during recovery. The proposed model assumes a monopolistic setup to understand how managers should advertise when they envision a crisis. But how should other competing brands react to the focal brand's crisis? Should they exploit its misfortune? To answer these questions, we apply differential game theory as in previous marketing research (e.g., Chintagunta and Vilcassim 1992, 1994; Jørgensen and Zaccour 2003; Naik et al. 2005; Bass et al. 2005; Erickson 2009a, b; Fruchter and Mantrala 2010).

4.6. Competition

Let A denote the focal brand anticipating the crisis, let B be the competing brand, and let $i = \{A, B\}$ index the two brands. Each brand's sales is affected by not only its own advertising $u_{ij}(S_A, S_B)$ and own sales but also by the market penetration of the competitor's brand as follows:

$$\frac{dS_{ij}}{dt} = \beta_{ij}\sqrt{u_{ij}}\sqrt{M - S_A - S_B} - \delta_{ij} S_i. \tag{18}$$

If a crisis occurs, brand A would lose its sales and increase the available market potential to B, thus fostering advertising competition. Each brand envisions this scenario and simultaneously determines its optimal advertising strategy to maximize the long-term profit. Suppressing t for notational clarity, the following proposition furnishes the optimal advertising strategies in Nash equilibrium.

Proposition 6 (Duopoly). *The Nash equilibrium feedback advertising strategies by the focal brand before and after the crisis, respectively, are given by*

$$u^*_{A1}(S_A, S_B) = (M - S_A - S_B)(0.5\beta_{A1}\lambda_{A1})^2 \quad \text{and}$$
$$u^*_{A2}(S_A, S_B) = (M - S_A - S_B)(0.5\beta_{A2}\lambda_{A2})^2.$$

Similarly, the Nash equilibrium feedback advertising strategies by the competing brand before and after the crisis, respectively, are given by

$$u^*_{B1}(S_A, S_B) = (M - S_A - S_B)(0.5\beta_{B1}\mu_{B1})^2 \quad \text{and}$$
$$u^*_{B2}(S_A, S_B) = (M - S_A - S_B)(0.5\beta_{B2}\mu_{B2})^2.$$

The parameters $(\lambda_{Aj}, \lambda_{Bj}, \mu_{Aj}, \mu_{Bj})'$ *are solutions to the algebraic equations given in the appendix.*

Proof. See the appendix.

Comparing this proposition with Proposition 2, we learn that the focal brand changes its advertising strategy in the presence of competing brands. Furthermore, B increases its advertising when it anticipates A's sales loss. In other words, the competing brand exploits the misfortunes of the focal brand. The above duopoly dynamic game can be generalized to oligopoly markets with any number of brands (see §5 for a triopoly market).

In sum, the normative analysis reveals that a crisis induces impatience (see Proposition 1); that the optimal advertising before (after) the crisis decreases (increases) when envisioning a crisis (see Proposition 3); that the optimal precrisis advertising decreases, but the postcrisis advertising increases as the anticipated loss in baseline sales increases (see Proposition 4); that the optimal advertising increases during recovery (see Proposition 5); and that the competing brands increase advertising to exploit the misfortunes of the focal brand (see Proposition 6).

We next develop a new method to estimate continuous-time models using discrete-time data that jointly specifies the sales dynamics and feedback strategies. We apply it to Ford Explorer's rollover recall data, obtain parameter estimates, and then compute the crisis likelihood and damage rate.

5. Empirical Analyses

5.1. Ford Explorer Recall

On August 7, 2000, Ford recalled the Explorer sports utility vehicle (SUV) because it consistently displayed tendencies to roll over (Govindaraj et al. 2004). This recall spurred over 200 lawsuits (*New York Times* 2002) and severed a century-old relationship between the Ford Motor Company and Firestone Tire Company (*New York Times* 2002). Explorer's rollover claims eventually cost Ford \$2.1 billion (Truett 2001).

In this study, we examine the pre-recall period from October 1996 to August 7, 2000, the recall period from August 7, 2000 to December 10, 2000, and the post-recall period from December 10, 2000 until June 30, 2002. A leading data supplier in the automotive industry provided data on pre- and postcrisis weekly sales and prices for major SUV brands sold in California. TNS Inc. furnished the advertising expenditures for the corresponding weeks. Before the recall, Explorer's average market share was 14.3%; this plummeted to 4.1% during the recall and then increased to 5.5% after the recall. Figure 3 displays the weekly sales (see panel A) and advertising (see panel B) before and after the recall. To account for competitive effects, we included the Jeep Cherokee and Toyota 4Runner brands, which constitute the top three brands in the SUV category. Table 1 presents the descriptive statistics on sales and advertising of the three brands and the SUV market size over time.

5.2. Continuous-Time Estimation

Let $i = \{A, B, C\}$ denote Ford Explorer, Jeep Cherokee, and Toyota 4Runner, respectively. Suppressing the regime subscript $j = \{1, 2\}$ for notational clarity (although each regime has its own parameter vector), we generalize (18) to a triopoly market to obtain the sales dynamics:

$$\begin{aligned}\dot{S}_A &= \beta_A\sqrt{u_A}\sqrt{M - S_A - S_B - S_C} - \delta_A S_A,\\ \dot{S}_B &= \beta_B\sqrt{u_B}\sqrt{M - S_A - S_B - S_C} - \delta_B S_B,\\ \dot{S}_C &= \beta_C\sqrt{u_C}\sqrt{M - S_A - S_B - S_C} - \delta_C S_C,\end{aligned} \tag{19}$$

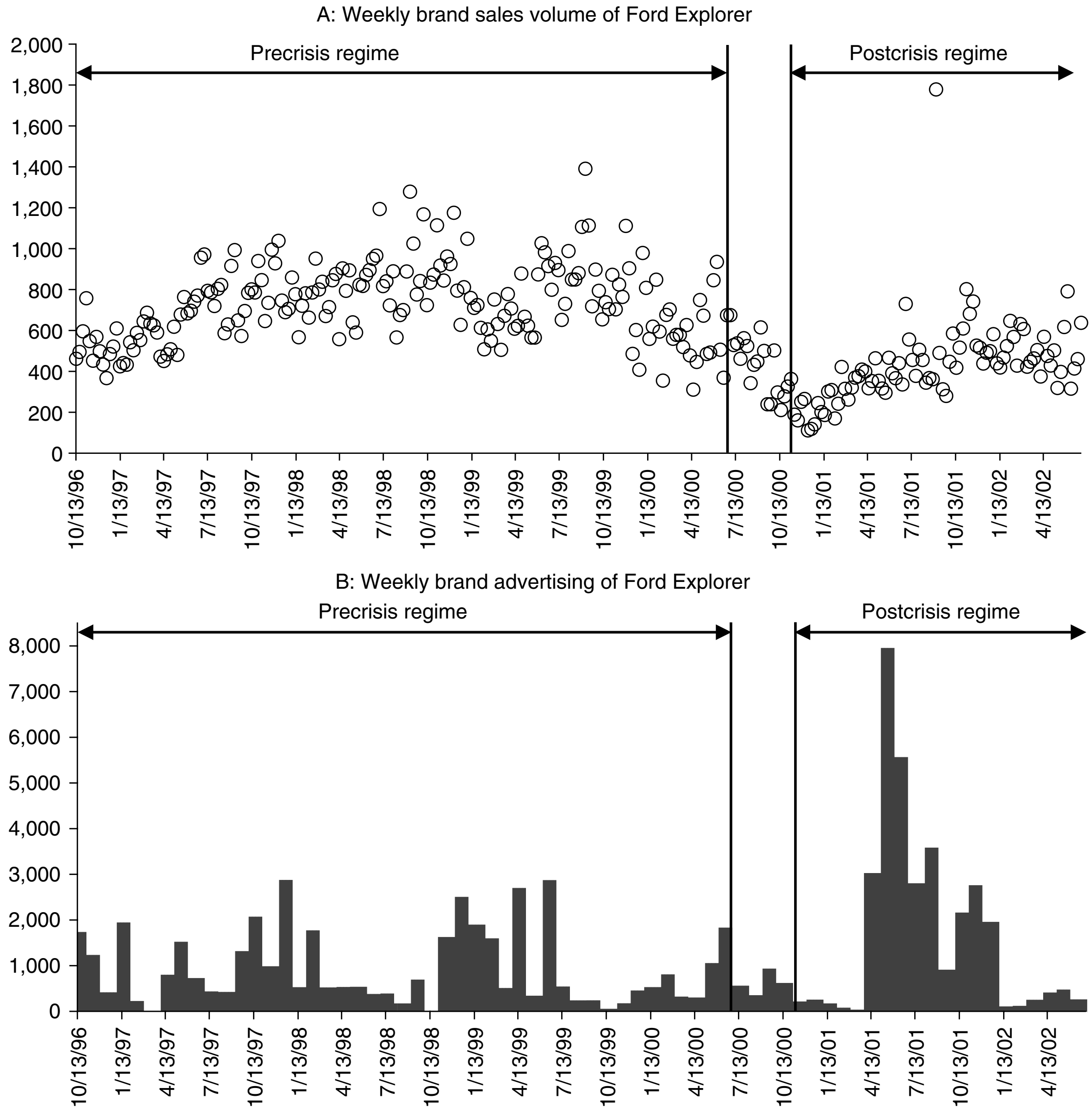
where the dot denotes the time derivative (i.e., $\dot{x} = dx/dt$). As in Proposition 6, we derive the three-brand feedback advertising strategies:

$$\begin{aligned}u_A &= (M - S_A - S_B - S_C)(0.5\beta_A\lambda_A)^2,\\ u_B &= (M - S_A - S_B - S_C)(0.5\beta_B\mu_B)^2,\\ u_C &= (M - S_A - S_B - S_C)(0.5\beta_C\gamma_C)^2.\end{aligned} \tag{20}$$

Then we substitute (20) into (19) to obtain the continuous-time sales dynamics:

$$\begin{bmatrix}\dot{S}_A\\ \dot{S}_B\\ \dot{S}_C\end{bmatrix} = \begin{bmatrix}-\delta_A - 0.5\beta_A^2\lambda_A & -0.5\beta_A^2\lambda_A & -0.5\beta_A^2\lambda_A\\ -0.5\beta_B^2\mu_B & -\delta_B - 0.5\beta_B^2\mu_B & -0.5\beta_B^2\mu_B\\ -0.5\beta_C^2\gamma_C & -0.5\beta_C^2\gamma_C & -\delta_C - 0.5\beta_C^2\gamma_C\end{bmatrix} \cdot \begin{bmatrix}S_A\\ S_B\\ S_C\end{bmatrix} + \begin{bmatrix}0.5\beta_A^2\lambda_A M\\ 0.5\beta_B^2\mu_B M\\ 0.5\beta_C^2\gamma_C M\end{bmatrix}. \tag{21}$$

Figure 3 Ford Explorer Sales and Brand Advertising



Equation (21) is a vector differential equation system, $\dot{S} = RS + r$, where $\dot{S} = (\dot{S}_A, \dot{S}_B, \dot{S}_C)'$.

The time t in (21) is continuously differentiable on any interval, $\underline{t} < t < \bar{t}$. However, observed data arrive at discrete points in time (e.g., weeks). That is, in the data series, the time parameter t_k is not continuously differentiable; rather, it takes discrete values in the integer set $\{1, 2, 3, \ldots, N\}$. To estimate continuous-time models using discrete-time data, Rao (1986) integrates the univariate sales dynamics and applies ordinary least squares to the resulting equation to obtain parameter estimates. Naik et al. (2005) extend this approach to multivariate sales dynamics with coupled differential equations and apply the Kalman filter to estimate model parameters. Below, we further extend their approach to estimate not only the multivariate sales dynamics but also the feedback advertising strategies simultaneously.

Table 1 Descriptive Statistics

	Precrisis		Postcrisis	
Variables	Avg	SD	Avg	SD
Sales (units/week)				
Ford Explorer	727	190	428	200
Jeep Cherokee	514	119	380	137
Toyota 4Runner	470	113	338	103
Advertising ($1,000/week)				
Ford Explorer	883	811	1,485	1,985
Jeep Cherokee	1,744	971	1,056	576
Toyota 4Runner	237	98	312	506
Market Size (units/week)	5,317	1,534	8,028	1,195

To this end, by applying the method of integrating factor, we integrate (21) over the interval $[t_{k-1}, t_k)$,

$$S_k - S_{k-1} = \int_{t_{k-1}}^{t_k} dS = \int_{t_{k-1}}^{t_k} \left(\frac{dS}{dt}\right) dt = \int_{t_{k-1}}^{t_k} (\dot{S})\, dt,$$

and we obtain the matrix equation,

$$S_k = Z_1 S_{k-1} + d_1, \tag{22}$$

where $Z_1 = \exp(R)$ is the transition matrix, $\exp(\cdot)$ is the *matrix* exponentiation function, and $d_1 = (\exp(R) - I)A^{-1}r$, with I being a 3×3 identity matrix. Unlike Rao (1986) and Naik et al. (2005), we do not assume constancy of advertising during $[t_{k-1}, t_k)$ because we incorporate advertising dynamics via the feedback strategies in (20). Also, note that the matrix exponentiation is not the same as the exponentiation of the elements of the matrix in general. The decay rate $\delta_i \in \Re^+$ and is not restricted to the unit interval as in discrete-time estimation. Consequently, we compute the usual carryover effects from the diagonal elements Z_1. Because the first element of Z_1 depends on $(\delta_A, \beta_A, \lambda_A)$, and likewise for the other diagonal elements, we learn that the carryover effects depend not only on the sales decay but also on ad effectiveness and brand profitability.

Next, we concatenate (22) with the feedback advertising strategies in (20) to obtain the state-space model:

$$\begin{bmatrix} S_k \\ u_k \end{bmatrix} = \begin{bmatrix} Z_1 & 0_{3\times 3} \\ Z_2 & 0_{3\times 3} \end{bmatrix} \begin{bmatrix} S_{k-1} \\ u_{k-1} \end{bmatrix} + \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} + \begin{bmatrix} \varepsilon_k \\ \nu_k \end{bmatrix}, \tag{23}$$

where $u_k = (u_{Ak}, u_{Bk}, u_{Ck})'$, $\varepsilon_k \sim \mathrm{N}(0, \sigma_\varepsilon^2 I)$, and $\nu_k \sim \mathrm{N}(0, \sigma_\nu^2 I)$ are sales and advertising transition noise, respectively, and

$$Z_2 = \begin{bmatrix} -0.25\beta_A^2\lambda_A^2 & -0.25\beta_A^2\lambda_A^2 & -0.25\beta_A^2\lambda_A^2 \\ -0.25\beta_B^2\mu_B^2 & -0.25\beta_B^2\mu_B^2 & -0.25\beta_B^2\mu_B^2 \\ -0.25\beta_C^2\gamma_C^2 & -0.25\beta_C^2\gamma_C^2 & -0.25\beta_C^2\gamma_C^2 \end{bmatrix}, \quad \text{and}$$

$$d_2 = \begin{bmatrix} 0.25\beta_A^2\lambda_A^2 M_{k-1} \\ 0.25\beta_B^2\mu_B^2 M_{k-1} \\ 0.25\beta_C^2\gamma_C^2 M_{k-1} \end{bmatrix}.$$

Denoting $\alpha_k = (S_k, u_k)'$ and expressing (23) as $\alpha_k = Z\alpha_{k-1} + d_k + \xi_k$, we link it to the observation equation $Y_k = \alpha_k + \xi_k$, where Y_k contains the observed sales and advertising data over time. We then apply the Kalman filter (see Naik et al. 1998 for details) to compute the evolving moments of α_k for $t_k = 1, \ldots, N$. Using those moments, we compute the log-likelihood of observing the joint sales sequence $\Omega_N = (Y_1, Y_2, \ldots, Y_N)'$, which is given by

$$L(\Theta; \Omega_N) = \sum_{k=1}^{N} \ln(p(Y_k \mid \Omega_{k-1})), \tag{24}$$

where $p(\cdot \mid \cdot)$ is the conditional density of Y_t given the sequence of sales up to the last period, Ω_{k-1}. The $K \times 1$ vector Θ contains the model parameters from each regime j, the error variances, and the initial mean of α_0. We maximize (24) with respect to Θ to obtain the maximum-likelihood estimates:

$$\hat{\theta} = \arg\max L(\Theta). \tag{25}$$

Then, we conduct robust statistical inference (White 1982). Specifically, if the model was correctly specified, we would obtain the standard errors from the square root of the diagonal elements of the inverse of the matrix:

$$Q = \left[-\frac{\partial^2 L(\Theta)}{\partial\Theta\,\partial\Theta'} \right]_{\Theta=\hat{\theta}}, \tag{26}$$

where Q equals the negative of the Hessian of L (Θ) evaluated at the estimated values $\hat{\theta}$. However, because the model is likely misspecified, we aim to make inferences robust to unknown misspecification errors. To obtain the robust standard errors, we apply the Huber–White correction known as the sandwich estimator (see White 1982):

$$\mathrm{Var}(\hat{\theta}) = Q^{-1} P Q^{-1}, \tag{27}$$

where P is a $K \times K$ matrix of the gradients of the log-likelihood function; that is, $P = G'G$, and G is $N \times K$ matrix obtained by stacking the $1 \times K$ vector of the gradient of the log-likelihood function for each of the N observations. For correctly specified models, $Q = P$, and so both Equations (26) and (27) yield exactly the same standard errors (as they should); otherwise, we use the robust standard errors given by the square root of the diagonal elements of (27). Thus, the robust standard errors safeguard us from the unknown forms of specification errors (White 1982).

Finally, we compare the proposed model with the specification proposed by Bass et al. (2005) that incorporates the cross effects of competing brand's advertising on own sales. Because their model does not contain the effects of crisis likelihood and damage rate, we first extend their model to incorporate those effects across both regimes. Second, we derive the feedback strategies for their extended model across both the pre- and postcrisis regimes. Third, by applying the above method, we estimate this extended model and present the resulting information criteria in Table 2. In panel A of Table 2, we observe that the extended Bass et al. (2005) model yields higher scores, indicating inferior predictive performance. Furthermore, both Akaike information criterion (AIC) and Bayesian information criterion (BIC) metrics indicate the same outcome, which enhances our confidence in the findings. Hence, we retain the proposed model based on model selection theory (e.g., Burnham and Anderson 2002, p. 70). Fourth, we investigate whether the *structure* of sales dynamics changes

Table 2 Model Comparison

A

Model	Precrisis AIC (BIC)	Postcrisis AIC (BIC)
M1 = Proposed model	4,948.6 (4,593.0)	2,396.4 (2,237.1)
M2 = Bass et al. (2005)	6,029.3 (5,705.4)	3,022.3 (2,890.5)

B

Models pre- and postcrisis	AIC (BIC)
(M1, M1)	7,345.1 (6,830.1)
(M2, M2)	9,051.6 (8,595.9)
(M1, M2)	7,970.9 (7,483.5)
(M2, M1)	8,425.7 (7,942.5)

Table 3 Continuous-Time Parameter Estimates

Parameters	Precrisis estimates (robust t-values)	Postcrisis estimates (robust t-values)
Ford Explorer		
Advertising effect, β_A	0.0023 (4.51)	0.0089 (3.36)
Sales decay, δ_A	0.1888 (5.09)	2.2967 (3.76)
Carryover effect, Z_{11}	0.8043	0.0899
Value function coefficient, λ_A	12,606	3,276
Jeep Cherokee		
Advertising effect, β_B	0.0084 (9.06)	0.0155 (28.51)
Sales decay, δ_B	1.3640 (8.61)	3.1506 (114.05)
Carryover effect, Z_{22}	0.2241	0.0481
Value function coefficient, μ_B	5,005	1,557
Toyota 4Runner		
Advertising effect, β_C	0.0284 (34.25)	0.0024 (2.60)
Sales decay, δ_C	1.7052 (47.87)	0.3578 (2.65)
Carryover effect, Z_{33}	0.1688	0.6765
Value function coefficient, γ_C	524	5,507

across regimes.[2] Specifically, we consider whether the proposed model (M1) and the Bass et al. (2005) model (M2) switches or continues across regimes. In other words, we estimate four pairs of dynamic models across the two regimes: (M1, M1), (M2, M2), (M1, M2), and (M2, M1). Moreover, each model has its own regime- and brand-specific parameters. Panel B of Table 2 reports the resulting information criteria. We find that the proposed model (M1, M1) attains the minimum score, indicating better predictive performance than that for the other three combinations. These results hold across both the AIC and BIC metrics, enhancing our confidence in the results. Hence, market data lend support to the proposed model.

In Table 3, we display the estimated parameters and their robust t-values for the proposed model. First, across all three brands, all the parameter estimates have expected signs and are significant at the 95% confidence level. Second, for both American brands, ad effectiveness increases and carryover effect decreases after the crisis. In contrast, for the Toyota 4Runner, ad effectiveness decreases after the crisis, whereas its carryover effect increases. The last two findings suggest the presence of compensatory effect: ad effectiveness and carryover effect move in opposite directions.

To understand these empirical results, consider the data from the National Highway Transportation Safety Administration (NHTSA) reported by Mayne et al. (2001). Specifically, Ford's share of all vehicles recalled was 32.1% (33.3%) in 2000 (1999), and that for Jeep was 28.2% (29.7%). Moreover, both Ford and Jeep were the two largest SUV brands. Indeed, "both brands…were perceived as similar by consumers in terms of quality, safety, performance, and aesthetics" (Vakratsas and Ma 2009, p. 29). Hence, according to the Feldman–Lynch theory, the common attribute of "Americanness" becomes diagnostic and accessible to consumers; thus the main competitor (Jeep) of the scandalized brand (Explorer) was "considered guilty by association" (Roehm and Tybout 2006, p. 366). Consequently, the enhanced media scrutiny focuses attention on American brands at the expense of the Japanese manufacturer, accounting for the ad effectiveness findings. Furthermore, NHTSA data report that Toyota's share of all vehicles recalled was as low as 0.03% (3.1%) in 2000 (1999). This greater (lower) reliability of Toyota (American brands) results in higher (lower) customer retention rates, accounting for the carryover effects.

In sum, our empirical results differ qualitatively from the uniform attenuation of model parameters documented by Van Heerde et al. (2007). Thus, we augment and complement the extant empirical literature. Specifically, we augment their findings by establishing the presence of compensatory shift in parameters. We complement the sparse extant literature by studying a different product category (cars versus peanut butter), where consumers' behavior differs because of infrequent purchase incidence and high price-to-income ratio.

5.3. Crisis Estimation

To characterize the crisis occurrence distribution, we first observe from the data that the damage rates are $\hat{\phi}_A = 0.346$ (Ford), $\hat{\phi}_B = 0.153$ (Jeep), and $\hat{\phi}_C = -0.123$ (Toyota). Then, using the estimated parameters $(\hat{\beta}_i, \hat{\delta}_i, \hat{\lambda}_A, \hat{\mu}_B, \hat{\gamma}_C)'$ for $i = \{A, B, C\}$, we compute the crisis likelihood as follows. The feedback strategies in (20) yield the analytical expressions for the coefficients of the value functions. Specifically, in each regime, we get three equations for each of the three brands (i.e., $3 \times 3 = 9$ per regime). Finally, for a given discount rate, we can solve these 18 equations

[2] We thank an anonymous reviewer for further extending this empirical analysis.

to find the unknown coefficients, which include the crisis likelihood. Accordingly, for $\rho = 6\%$ per annum, we compute $\hat{\chi} = 0.476$ per annum. Based on §3.2, the density function of the random crisis timing T is $f(t) = \chi \exp(-\chi t)$. Hence, the expected crisis time for Ford Explorer is $E[T] = \chi^{-1} \approx 2.1$ years. In addition, the percentiles of T are given by $F^{-1}(p; \chi) = -\ln(1-p)/\chi$, and so the 95th percentile $T_{0.95} = (-\ln(1-0.95))/0.476 = 6.3$ years. Applying this procedure to different discount rates, we find a robust range for the expected crisis timing. Specifically, we doubled the discount rate to $\rho = 12\%$ per annum to get $E[T] = 1/0.403 \approx 2.5$ years; we tripled it to $\rho = 18\%$ per annum to get $E[T] = 1/0.331 \approx 3.0$ years. Thus, Ford Explorer should expect a crisis in two to three years.

We close this section by furnishing the confidence interval for the pre- and postcrisis optimal advertising. To this end, we apply Krinsky and Robb's (1986) approach to obtain confidence intervals. Specifically, we randomly draw 10,000 points from the asymptotic distribution of the estimated parameters and then evaluate (20) for the optimal pre- and postcrisis advertising. The resulting 2.5th and 97.5th percentiles cover the 95% confidence interval. Table 4 reports the confidence intervals. We observe that the average ad spending for each brand in each regime (see Table 1) lies well within their corresponding intervals. Although overspending exists before the crisis, actual advertising after the crisis is 1.7% above the optimal expenditures for the Ford Explorer, 4.4% for the Jeep Cherokee, and 1.9% for the Toyota 4Runner. Thus, the optimal decisions comport with the managers' actions.

Table 4 Observed vs. Optimal Advertising

	Precrisis ($000)	Postcrisis ($000)
Ford Explorer		
95% confidence interval for optimal advertising	(315, 1,386)	(1,098, 1,903)
Optimal advertising	717	1,460
(Observed − optimal)/optimal advertising	23.2%	1.7%
Jeep Cherokee		
95% confidence interval for optimal advertising	(663, 2,912)	(760, 1,317)
Optimal advertising	1,507	1,011
(Observed − optimal)/optimal advertising	15.7%	4.4%
Toyota 4Runner		
95% confidence interval for optimal advertising	(84, 368)	(230, 399)
Optimal advertising	190	306
(Observed − optimal)/optimal advertising	24.7%	1.9%

6. Discussion

Here, we discuss the size-dependent insulation effect, investment in quality to reduce the crisis likelihood, and pricing strategy.

6.1. Size-Dependent Insulation Effect

In the proposed model, we assumed the sales damage is independent of brand's size. But suppose larger brands suffer a smaller loss in baseline sales because of their brand equity.[3] To account for this size-dependent insulation effect, we extend the damage function $\phi(S) = \phi_0 + \phi_1/S$ (with $S \geq 1$). Then we obtain the following.

Proposition 7 (Insulation Effect). *The optimal pre- and postcrisis feedback advertising strategies are* $u_1^*(S(t)) = (M(t) - S(t))(0.5\beta_1\lambda_1)^2$ *and* $u_2^*(S(t)) = (M(t) - S(t))(0.5\beta_2\lambda_2)^2$, *respectively, where*

$$\lambda_1 = \frac{-2(\rho+\delta_1+\chi)+2\sqrt{(\rho+\delta_1+\chi)^2+(m_1+\chi(1-\phi_0)\lambda_2)\beta_1^2}}{\beta_1^2},$$

and

$$\lambda_2 = \frac{-2(\rho+\delta_2)+2\sqrt{(\rho+\delta_2)^2+m_2\beta_2^2}}{\beta_2^2}.$$

Proof. See the appendix.

Although Proposition 7 appears similar to Proposition 2, it yields two new insights. First, optimal ad spending under a size-dependent insulation effect exceeds that under a size-independent insulation effect. Thus, intuitively, when the insulation effect depends on the brand's size, the brand manager seeks to grow sales by spending more on advertising, thereby reducing the damage rate. Second, Proposition 7 reveals that the optimal precrisis ad spending does not depend on ϕ_1, which is the coefficient for size-dependent effect. Why? The answer is that brands have incentive to grow and reduce the damage rate; consequently, managers invest in advertising to attain the largest profitable size and the smallest damage rate (which is ϕ_0).

6.2. Investments in Quality

If product-harm crisis is brewing, should managers invest in improving the product quality? Chen et al. (2009) find that the firm's market value reduces substantially under the proactive strategy (i.e., recall the defective products before a crisis occurs) more so than it does under the reactive strategy (i.e., wait for the crisis to occur before initiating the recall). In other words, investors impose a larger financial penalty for the proactive strategy because they infer that the impending crisis must be severe enough for managers to recall the product sold. So investment to improve

[3] We thank an anonymous reviewer for this suggestion.

the quality of products sold may not be financially prudent.

However, should managers invest in quality improvement for future production runs? That is, can managers reduce the crisis probability by investing resources to enhance product quality? Let $\{\underline{v}, \bar{v}\}$ denote low or high investment in quality enhancement, which results in a high or low crisis likelihood $\{\bar{\chi}, \underline{\chi}\}$, respectively. Then, in the appendix, we derive the new result that a brand should increase both ad spending and quality investments to lower the crisis likelihood only if its sales exceed a certain threshold; otherwise, it reduces both ad spending and quality investment, and it bears a higher risk of crisis.

6.3. Optimal Pricing Strategy

To set optimal prices, managers can utilize the framework developed by Bass et al. (2005, p. 560), which can be extended for the crisis case. To illustrate briefly, the Hamilton–Jacobi–Bellman equations for the two competing brands A and B in the precrisis regime are

$$\begin{aligned}(\rho+\chi)&V_i(S_A, S_B)\\ &= \max_{u_{i1}, p_{i1}}\bigg\{(p_{i1}-c_i)(1-a_{i1}p_{i1}+b_{i1}p_{3-i,1})S_i - u_{i1}\\ &\quad + \chi V_i((1-\phi_A)S_A, (1-\phi_B)S_B)\\ &\quad + \sum_i \frac{\partial W_i}{\partial S_i}(\beta_{i1}\sqrt{u_{i1}}\sqrt{M-S_A-S_B}-\delta_{i1}S_i)\bigg\},\end{aligned}$$

where $(1-a_{i1}p_{i1}+b_{i1}p_{3-i;1})$ captures the sales shrinkage as a result of price competition (see Footnote 2 in Bass et al. 2005). As the appendix shows, the optimal prices in regime j are given by

$$p^*_{Aj} = \frac{b_{Aj}+a_{Bj}(2+2a_{Aj}c_A+b_{Aj}c_B)}{4a_{Aj}a_{Bj}-b_{Aj}b_{Bj}} \quad \text{and}$$

$$p^*_{Bj} = \frac{b_{Aj}+a_{Aj}(2+2a_{Bj}c_B+b_{Bj}c_A)}{4a_{Aj}a_{Bj}-b_{Aj}b_{Bj}}.$$

The feedback advertising strategies are similar to those in Proposition 6, but the associated algebraic equations (see (73)–(76) and (80)–(83) in the appendix) depend on a_{i1} and b_{i1}. Analyzing this equilibrium, we learn that intensified price competition reduces optimal advertising because long-term profitability diminishes.

7. Conclusions

A French adage, "*gérer, c'est prévoir*," tells us that "managing is envisioning." Hence, brand managers should envision the unexpected. They can estimate the crisis likelihood and damage rate as illustrated via the analyses of Ford Explorer's sales advertising data. Specifically, the expected crisis time ranges from two to three years, a crisis occurs within six years at the 95% confidence level, and it damages the baseline sales by 35%.

Nonetheless, the next crisis will strike at a random time, and hence managers should envision the risk ex ante and incorporate it in their decision making. One way to manage this risk is to reduce precrisis ad spending and increase postcrisis ad spending to recover lost sales. In competitive markets, nonfocal brands should increase advertising to exploit the sales lost by the focal brand. Finally, managers can reduce the crisis likelihood by investing in the quality of future products.

We contribute not only new substantive and empirical results but also methodological and modeling innovations. Specifically, a novel substantive insight is that brand managers should change their precrisis advertising decisions even though a crisis affects the postcrisis parameters. Why? The answer is that forward-looking managers anticipate a reduction in baseline sales and/or long-term profitability after the crisis. This anticipation enhances impatience, which reduces precrisis advertising and thereby induces a double whammy. First, the sales level at the time of the crisis is lower because of reduced precrisis advertising; additionally, the brand loses customers when the crisis occurs. To recover this sales loss resulting from double adverse effects, the optimal response is to increase advertising after the crisis. In a nutshell, managers should reduce ad spending before the crisis but advertise more when needed after it occurs.

A novel empirical insight is that the parameters change in a compensatory manner: ad effectiveness increases while carryover effect decreases (or vice versa). Because of media scrutiny, customers focus attention on focal or similarly categorized brands, which enhances ad effectiveness. However, media scrutiny makes salient their lower reliability, hurting customer retention and carryover effects. This parametric shift differs qualitatively from the uniform attenuation documented by Van Heerde et al. (2007). Collectively, our findings complement the sparse empirical literature on the impact of product-harm crisis.

A novel methodological contribution is the development of a method to estimate continuous-time models using discrete-time data that jointly specifies the sales dynamics and feedback strategies. We learn that the decay rate is not restricted to the unit interval as in discrete-time estimation and that the carryover effect depends on not only the decay rate but also ad effectiveness and brand profitability.

Finally, a novel modeling innovation is the nature of uncertainty from rare catastrophic events, which are large and discrete rather than small and continuous as formulated previously (e.g., Raman and

Chatterjee 1995, Raman and Naik 2004, Prasad and Sethi 2004). The resulting random stopping problem and its solution approach in §3 are new to marketing science. We encourage researchers to adapt this framework to solve other open marketing problems (e.g., optimal decisions in the presence of competitive entry or technological innovation).

We close by identifying two important topics for further investigation. We restricted the scope of our normative analyses to product-harm crisis because of our empirical application. However, the nature of the random timing of crisis and its impact on profitability (see §3) permit a broader interpretation of what constitutes a crisis. For example, financial crisis or recessions or wars are also catastrophic events beyond management's control. We encourage future researchers to use the proposed framework and appropriate data to investigate the marketing implications of such general crises. Second, in this study, the value function after the crisis is $W_A = 3{,}276\ S_A - 59\ S_B - 435\ S_C + 2.5\times 10^7$, which shows that the cash flows from profits remain positive for the prevailing own and competitors' sales. Although bankruptcy does not arise in our case, we encourage future research to investigate this phenomenon using control-theoretic models (see Sethi 1997) and thereby improve the theory and practice of crisis management.

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Appendix

Proof of Equation (9). To evaluate I_1 in Equation (8), we apply the integration by parts, $\int U\,dV = UV - \int V\,dU$, where $U(t) = \int_0^t e^{-\rho s}\pi(s)\,ds$ and $V(t) = -e^{-\chi t}$ so that $dU = e^{-\rho t}\pi(t)dt$ and $dV = \chi e^{-\chi t}dt$. Making the appropriate substitutions, we obtain

$$\begin{aligned} I_1 &= \int_0^\infty \Big[\int_0^t e^{-\rho s}\pi(s)\,ds\Big]\chi e^{-\chi t}\,dt \\ &= (-U(t)e^{-\chi t})|_0^\infty - \int_0^\infty (-e^{-\chi t})(e^{-\rho t}\pi(t)\,dt) \\ &= 0 + \int_0^\infty e^{-(\rho+\chi)t}\pi(t)\,dt, \end{aligned}$$

where the first term vanishes because $\lim_{t\to\infty} e^{-\chi t} = 0$ and $U(0) = \int_0^0 e^{-\rho s}\pi(s)\,ds = 0$.

We next prove the Propositions 2–7 and offer derivations to support the other results.

Proof of Proposition 2 (Optimal Advertising Strategies). We suppress t for notational clarity. Applying backward induction, we first solve the problem in the postcrisis regime. The Hamilton–Jacobi–Bellman (HJB) equation is

$$\rho W(S) = \max_{u_2}\Big\{m_2 S - u_2 + \frac{\partial W}{\partial S}(\beta_2\sqrt{u_2}\sqrt{M-S} - \delta_2 S)\Big\}. \quad (28)$$

By differentiating the parenthetical expression in (28) with respect to (w.r.t.) u_2, we obtain the first-order condition (FOC):

$$\begin{aligned} &-1 + \frac{\beta_2\sqrt{M-S}}{2\sqrt{u_2}}\partial W/\partial S = 0 \\ &\Rightarrow\ u_2^*(S) = (M-S)(0.5\beta_2\partial W/\partial S)^2. \end{aligned} \quad (29)$$

We substitute (29) in (28) to obtain an ordinary differential equation (ODE) in $W(S)$. To solve this ODE, we apply the method of underdetermined coefficients. Specifically, we first conjecture

$$W(S) = \lambda_2 S + \lambda_{02} \quad (30)$$

and then identify the coefficients as

$$\lambda_2 = \frac{-2(\rho+\delta_2) + 2\sqrt{(\rho+\delta_2)^2 + m_2\beta_2^2}}{\beta_2^2}, \quad \text{and} \quad (31)$$

$$\lambda_{02} = \frac{M(\beta_2\lambda_2)^2}{4\rho}. \quad (32)$$

Hence, the postcrisis feedback strategy is $u_2^*(S(t)) = (M(t) - S(t))(0.5\beta_2\lambda_2)^2$, proving one-half of Proposition 2.

In the precrisis regime, the HJB equation is

$$\begin{aligned} (\rho+\chi)V(S) = \max_{u_1}\Big\{&m_1 S - u_1 + \chi W((1-\phi))S \\ &+ \frac{\partial V}{\partial S}(\beta_1\sqrt{u_1}\sqrt{M-S} - \delta_1 S)\Big\}. \end{aligned} \quad (33)$$

We obtain the FOC w.r.t. u_1:

$$\begin{aligned} &-1 + \frac{\beta_1\sqrt{M-S}}{2\sqrt{u_1}}\partial V/\partial S = 0 \\ &\Rightarrow\ u_1^*(S) = (M-S)(0.5\beta_1\partial V/\partial S)^2. \end{aligned} \quad (34)$$

By substituting (34) in (33), we obtain an ODE in $V(S)$ whose solution is

$$V(S) = \lambda_1 S + \lambda_{01}, \quad (35)$$

where

$$\lambda_1 = \frac{-2(\rho+\delta_1+\chi) + 2\sqrt{(\rho+\delta_1+\chi)^2 + (m_1+\chi(1-\phi)\lambda_2)\beta_1^2}}{\beta_1^2}, \quad (36)$$

and

$$\lambda_{01} = \frac{M(\beta_1\lambda_1)^2}{4(\rho+\chi)} + \frac{\chi\lambda_{02}}{(\rho+\chi)}. \quad (37)$$

The precrisis feedback strategy is $u_1^*(S(t)) = (M(t) - S(t))(0.5\beta_1\lambda_1)^2$, thus completing the proof. □

Proof of Proposition 3 (Crisis Likelihood Effects). We differentiate $u_1^*(S)$ with respect to χ. The $\text{sign}(\partial u_1^*/\partial\chi)$ is negative if $\lambda_2 < k = 2[-(\rho+\delta_1) + \sqrt{(\rho+\delta_1)^2 + m_1\beta_1^2}]/((1-\phi)\beta_1^2)$ and positive otherwise. Furthermore,

$$\frac{2[-(\rho+\delta_1)+\sqrt{(\rho+\delta_1)^2+m_1\beta_1^2}]}{((1-\phi)\beta_1^2)} > \frac{2[-(\rho+\delta_1)+\sqrt{(\rho+\delta_1)^2+m_1\beta_1^2}]}{\beta_1^2},$$

which equals the long-term profit in the absence of crisis. Since a crisis diminishes the brand's long-term profit,

$$\frac{2[-(\rho+\delta_1)+\sqrt{(\rho+\delta_1)^2+m_1\beta_1^2}]}{\beta_1^2} > \frac{2[-(\rho+\delta_2)+\sqrt{(\rho+\delta_2)^2+m_2\beta_2^2}]}{\beta_2^2} = \lambda_2.$$

Hence $\lambda_2 < k$, which implies that $\partial u_1^*/\partial\chi < 0$.

As χ increases, u_1^* becomes smaller for every t, so $S(T^-)$ decreases, which, in turn, suppresses $S(T^+)$ because of Equation (4). Consequently, the feedback advertising u_2^* increases (because of Proposition 2). Hence $du_2^*/d\chi > 0$. □

Proof of Proposition 4 (Damage Effects). We differentiate $\partial u_1^*/\partial\phi = -(\chi\lambda_2)/\sqrt{(\rho+\delta_1)^2 + (m_1+\chi(1-\phi))\beta_1^2}$, which shows that $\text{sign}(\partial u_1^*/\partial\phi) < 0$.

As ϕ increases, u_1^* becomes smaller for every t, which reduces $S(T^-)$. In addition, ϕ directly lowers $S(T^+)$ because of Equation (4). These dual adverse effects suppress the sales trajectory $S(t)$ for $t > T$. Consequently, the feedback advertising via Proposition 2 increases u_2^* as ϕ increases. Hence $du_2^*/d\phi > 0$. □

Proof of Proposition 5 (Recovery). The HJB equation for the postcrisis regime is

$$(\rho+\omega)W(S) = \max_{u_2}\left\{m_2S - u_2 + \omega V((1+\theta)S) + \frac{\partial W}{\partial S}\left(\beta_2\sqrt{u_2}\sqrt{M-S} - \delta_2 S\right)\right\} \quad (38)$$

and for the precrisis regime is

$$(\rho+\chi)V(S) = \max_{u_1}\left\{m_1S - u_1 + \chi W((1-\phi)S) + \frac{\partial V}{\partial S}\left(\beta_1\sqrt{u_1}\sqrt{M-S} - \delta_1 S\right)\right\}. \quad (39)$$

Differentiating (38) and (39), we get the following FOCs: $u_1^*(S) = (M-S)(0.5\beta_1\partial V/\partial S)^2$ and $u_2^*(S) = (M-S)(0.5\beta_2\partial W/\partial S)^2$. Substituting the FOCs back into (38) and (39), we obtain ODEs in $V(S)$ and $W(S)$, whose solutions are given by $V(S) = \lambda_1 S + \lambda_{01}$ and $W(S) = \lambda_2 S + \lambda_{02}$, where $(\lambda_1, \lambda_2, \lambda_{01}, \lambda_{02})$ are obtained from the following algebraic equations:

$$\lambda_1 = \frac{-2(\rho+\delta_1+\chi)+2\sqrt{(\rho+\delta_1+\chi)^2+(m_1+\chi(1-\phi)\lambda_2)\beta_1^2}}{\beta_1^2}, \quad (40)$$

$$\lambda_2 = \frac{-2(\rho+\delta_2+\omega)+2\sqrt{(\rho+\delta_2+\omega)^2+(m_2+\omega(1+\theta)\lambda_1)\beta_2^2}}{\beta_2^2}, \quad (41)$$

$$\lambda_{01} = \frac{M(\beta_1\lambda_1)^2}{4(\rho+\chi)} + \frac{\chi\lambda_{02}}{(\rho+\chi)}, \quad \text{and} \quad (42)$$

$$\lambda_{02} = \frac{M(\beta_2\lambda_2)^2}{4(\rho+\theta)} + \frac{\theta\lambda_{01}}{(\rho+\theta)}. \quad (43)$$

Thus, suppressing t, the feedback strategies are $u_1^*(S) = (M-S)(0.5\beta_1\lambda_1)^2$ and $u_2^*(S) = (M-S)(0.5\beta_2\lambda_2)^2$. □

Proof of Proposition 6 (Duopoly). In the second regime, the HJB equation for brand $i = \{A, B\}$ is

$$\rho W_i(S_A, S_B) = \max_{u_{i2}}\left\{m_{i2}S_i - u_{i2} + \sum_i \frac{\partial W_i}{\partial S_i}\left(\beta_{i2}\sqrt{u_{i2}}\sqrt{M-S_A-S_B} - \delta_{i2}S_i\right)\right\}. \quad (44)$$

By differentiating (44) w.r.t. u_{i2}, we obtain the FOC:

$$-1 + \frac{\beta_{i2}\sqrt{M-S_A-S_B}}{2\sqrt{u_{i2}}}\partial W_i/\partial S_i = 0$$
$$\Rightarrow \quad u_{i2}^*(S_A, S_B) = (M - S_A - S_B)(0.5\beta_{i2}\partial W_i/\partial S_i)^2. \quad (45)$$

We substitute (45) in (44) to obtain a system of ODEs in $W_i(S_A, S_B)$ whose solutions are

$$\begin{bmatrix} W_A(S_A, S_B) \\ W_B(S_A, S_B) \end{bmatrix} = \begin{bmatrix} \lambda_{A2} & \lambda_{B2} \\ \mu_{A2} & \mu_{B2} \end{bmatrix} \begin{bmatrix} S_A \\ S_B \end{bmatrix} + \begin{bmatrix} \lambda_{02} \\ \mu_{02} \end{bmatrix}, \quad (46)$$

where the coefficients are obtained from the following algebraic equations:

$$\rho\lambda_{A2} = \tfrac{1}{4}(4m_{A2} - 4\delta_{A2}\lambda_{A2} - \beta_{A2}^2\lambda_{A2}^2 - 2\beta_{B2}^2\mu_{B2}\lambda_{B2}), \quad (47)$$
$$\rho\lambda_{B2} = \tfrac{1}{4}(-\beta_{A2}^2\lambda_{A2}^2 - 4\delta_{B2}\lambda_{B2} - 2\beta_{B2}^2\mu_{B2}\lambda_{B2}), \quad (48)$$
$$\rho\mu_{A2} = \tfrac{1}{4}(-\beta_{B2}^2\mu_{B2}^2 - 4\delta_{A2}\mu_{A2} - 2\beta_{A2}^2\mu_{A2}\lambda_{A2}), \quad \text{and} \quad (49)$$
$$\rho\mu_{B2} = \tfrac{1}{4}(4m_{B2} - 4\delta_{B2}\mu_{B2} - \beta_{B2}^2\mu_{B2}^2 - 2\beta_{A2}^2\lambda_{A2}\mu_{A2}). \quad (50)$$

Thus, the postcrisis optimal feedback strategies are

$$u_{A2}^*(S_A(t), S_B(t)) = (M(t) - S_A(t) - S_B(t))(0.5\beta_{A2}\lambda_{A2})^2 \quad \text{and}$$
$$u_{B2}^*(S_A(t), S_B(t)) = (M(t) - S_A(t) - S_B(t))(0.5\beta_{B2}\mu_{B2})^2.$$

In the precrisis regime, the HJB equations are

$$(\rho+\chi)V_i(S_A, S_B) = \max_{u_{i1}}\left\{m_{i1}S_i - u_{i1} + \chi W_i((1-\phi_A)S_A, (1-\phi_B)S_B) + \sum_i \frac{\partial V_i}{\partial S_i}\left(\beta_{i1}\sqrt{u_{i1}}\sqrt{M-S_A-S_B} - \delta_{i1}S_i\right)\right\}. \quad (51)$$

By differentiating (51) w.r.t. u_{i1}, we obtain the FOCs:

$$-1+\frac{\beta_{i1}\sqrt{M-S_A-S_B}}{2\sqrt{u_{i1}}}\partial V_i/\partial S_i=0$$
$$\Rightarrow\quad u_{i1}^*(S_A,S_B)=(M-S_A-S_B)(0.5\beta_{i1}\partial V_i/\partial S_i)^2. \quad (52)$$

We substitute (52) in (51) to obtain a system of ODEs in $V_i(S_A,S_B)$ whose solutions are

$$\begin{bmatrix} V_A(S_A,S_B) \\ V_B(S_A,S_B) \end{bmatrix}=\begin{bmatrix} \lambda_{A1} & \lambda_{B1} \\ \mu_{A1} & \mu_{B1} \end{bmatrix}\begin{bmatrix} S_A \\ S_B \end{bmatrix}+\begin{bmatrix} \lambda_{01} \\ \mu_{01} \end{bmatrix}, \quad (53)$$

where the coefficients are obtained from the algebraic equations:

$$(\rho+\chi)\lambda_{A1}=\tfrac{1}{4}(4m_{A1}-4\delta_{A1}\lambda_{A1}-\beta_{A1}^2\lambda_{A1}^2 -2\beta_{B1}^2\mu_{B1}\lambda_{B1}+4\chi(1-\phi_A)\lambda_{A2}), \quad (54)$$

$$(\rho+\chi)\lambda_{B1}=\tfrac{1}{4}(-\beta_{A1}^2\lambda_{A1}^2-4\delta_{B1}\lambda_{B1}-2\beta_{B1}^2\mu_{B1}\lambda_{B1} +4\chi(1-\phi_B)\lambda_{B2}), \quad (55)$$

$$(\rho+\chi)\mu_{A1}=\tfrac{1}{4}(-\beta_{B1}^2\mu_{B1}^2-4\delta_{A1}\mu_{A1}-2\beta_{A1}^2\mu_{A1}\lambda_{A1} +4\chi(1-\phi_A)\mu_{A2}),\quad \text{and} \quad (56)$$

$$(\rho+\chi)\mu_{B1}=\tfrac{1}{4}(4m_{B1}-4\delta_{B1}\mu_{B1}-\beta_{B1}^2\mu_{B1}^2 -2\beta_{A1}^2\lambda_{A1}\mu_{A1}+4\chi(1-\phi_B)\mu_{B2}). \quad (57)$$

Thus, the precrisis optimal advertising strategies are

$$u_{A1}^*(S_A(t),S_B(t))=(M(t)-S_A(t)-S_B(t))(0.5\beta_{A1}\lambda_{A1})^2 \quad \text{and}$$
$$u_{B1}^*(S_A(t),S_B(t))=(M(t)-S_A(t)-S_B(t))(0.5\beta_{B1}\mu_{B1})^2. \quad \square$$

Proof for Proposition 7 (Insulation Effect). The HJB equation for the brand reads

$$(\rho+\chi)V(S)=\max_{u_1}\left\{m_1S-u_1+\chi W\left(\left(1-\left(\phi_0+\frac{\phi_1}{S}\right)\right)S\right) +\frac{\partial V}{\partial S}\left(\beta_1\sqrt{u_1}\sqrt{M-S}-\delta_1 S\right)\right\}. \quad (58)$$

Next, we solve for the precrisis optimal feedback strategies with a size-dependent damage rate. Differentiating (58) w.r.t. u_1, inserting the FOC into (58), and solving the resulting ODE, we obtain the optimal feedback strategy

$$u_1^*(S)=(M-S)(0.5\beta_1\partial W_1/\partial S)^2 \quad (59)$$

and the value function

$$W_1(S)=\lambda_1S+\lambda_{01}, \quad (60)$$

where

$$\lambda_1=\frac{-2(\rho+\delta_1+\chi)+2\sqrt{(\rho+\delta_1+\chi)^2+(m_1+\chi(1-\phi_0)\lambda_2)\beta_1^2}}{\beta_1^2}, \quad (61)$$

and

$$\lambda_{01}=\frac{M(\beta_1\lambda_1)^2}{4(\rho+\chi)}+\frac{\chi(\lambda_{02}-\phi_1\lambda_2)}{(\rho+\chi)}. \quad \square \quad (62)$$

Proof for §6.2 (Investment in Quality). If the manager chooses the low investment $\underline{v}$ in quality, the HJB equation is

$$(\rho+\bar{\chi})V_L(S)=\max_{u_1}\left\{m_1S-u_1-\underline{v}+\bar{\chi}W((1-\phi)S) +\frac{\partial V_L}{\partial S}\left(\beta_1\sqrt{u_1}\sqrt{M-S}-\delta_1S\right)\right\}. \quad (63)$$

If he or she chooses the high investment $\bar{v}$, the HJB equation is

$$(\rho+\underline{\chi})V_H(S)=\max_{u_1}\left\{m_1S-u_1-\bar{v}+\underline{\chi}W((1-\phi)S) +\frac{\partial V_H}{\partial S}\left(\beta_1\sqrt{u_1}\sqrt{M-S}-\delta_1S\right)\right\}. \quad (64)$$

Deriving the FOCs and solving the resulting ODEs, we find that, under the high investment $\bar{v}$, the value function is

$$V_H(S)=\lambda_{1H}S+\frac{\underline{\chi}\lambda_{02}-\bar{v}}{\rho+\underline{\chi}}+\frac{M(\beta_1\lambda_{1H})^2}{4(\rho+\underline{\chi})}, \quad (65)$$

where

$$\lambda_{1H}=\frac{-2(\rho+\delta_1+\underline{\chi})+2\sqrt{(\rho+\delta_1+\underline{\chi})^2+(m_1+\underline{\chi}(1-\phi_0)\lambda_2)\beta_1^2}}{\beta_1^2}, \quad (66)$$

and

$$\lambda_2=\frac{-2(\rho+\delta_2)+2\sqrt{(\rho+\delta_2)^2+m_2\beta_2^2}}{\beta_2^2}. \quad (67)$$

For the low investment $\underline{v}$, the value function is

$$V_L(S)=\lambda_{1L}S+\frac{\bar{\chi}\lambda_{02}-\underline{v}}{\rho+\bar{\chi}}+\frac{M(\beta_1\lambda_{1L})^2}{4(\rho+\bar{\chi})}, \quad (68)$$

where

$$\lambda_{1L}=\frac{-2(\rho+\delta_1+\bar{\chi})+2\sqrt{(\rho+\delta_1+\bar{\chi})^2+(m_1+\bar{\chi}(1-\phi_0)\lambda_2)\beta_1^2}}{\beta_1^2}. \quad (69)$$

It follows from Proposition 3 that $\partial\mu_1/\partial\chi<0$ because $\partial\lambda_1/\partial\chi<0$. That is, λ is a decreasing function of χ. Hence, $\lambda_{1H}>\lambda_{1L}$ since $\bar{\chi}<\underline{\chi}$. Given the different slopes of the value function, the manager chooses $\bar{v}$ if $V_H(S)>V_L(S)$ and $\underline{v}$ otherwise. The crossover of the two value functions yields the sales threshold:

$$S^*=\Big(0.5\beta_1^2((\underline{\chi}\lambda_{02}-\bar{v})/(\rho+\underline{\chi})-(\bar{\chi}\lambda_{02}-\underline{v})/(\rho+\bar{\chi}) +M(\beta_1\lambda_{1H})^2/4(\rho+\underline{\chi})-M(\beta_1\lambda_{1L})^2/4(\rho+\bar{\chi}))\Big) \cdot\Big(\bar{\chi}-\underline{\chi}+\sqrt{(\rho+\delta_1+\underline{\chi})^2+(m_1+\underline{\chi}(1-\phi_0)\lambda_2)\beta_1^2} -\sqrt{(\rho+\delta_1+\bar{\chi})^2+(m_1+\bar{\chi}(1-\phi_0)\lambda_2)\beta_1^2}\Big)^{-1}. \quad \square$$

Proof for §6.3 (Price and Advertising Competition). In the postcrisis regime, the HJB equations for the brands $i=\{A, B\}$ are

$$\rho W_i(S_A, S_B) = \max_{u_{i2}, p_{i2}} \left\{ (p_{i2}-c_i)(1-a_{i2}p_{i2}+b_{i2}p_{3-i,2})S_i - u_{i2} + \sum_i \frac{\partial W_i}{\partial S_i}\left(\beta_{i2}\sqrt{u_{i2}}\sqrt{M-S_A-S_B}-\delta_{i2}S_i\right)\right\}. \quad (70)$$

By differentiating (70) w.r.t. both the controls u_{i2} and p_{i2}, we obtain the FOCs:

$$-1+\frac{\beta_{i2}\sqrt{M-S_A-S_B}}{2\sqrt{u_{i2}}}\partial W_i/\partial S_i = 0 \quad (71)$$

$$\Rightarrow \quad u_{i2}^*(S_A, S_B) = (M-S_A-S_B)(0.5\beta_{i2}\partial W_i/\partial S_i)^2,$$

and $p_{i2}=(1+a_{i2}c_i+b_{i2}p_{3-i,2})/2a_{i2}$, which leads to the optimal prices:

$$p_{A2}^* = \frac{b_{A2}+a_{B2}(2+2a_{A2}c_A+b_{A2}c_B)}{4a_{A2}a_{B2}-b_{A2}b_{B2}} \quad \text{and} \quad p_{B2}^* = \frac{b_{A2}+a_{A2}(2+2a_{B2}c_B+b_{B2}c_A)}{4a_{A2}a_{B2}-b_{A2}b_{B2}}. \quad (72)$$

We substitute (71) and (72) in (70) to obtain the system of ODEs in $W_i(S_A, S_B)$ whose solutions are given by

$$\begin{bmatrix} W_A(S_A, S_B) \\ W_B(S_A, S_B) \end{bmatrix} = \begin{bmatrix} \lambda_{A2} & \lambda_{B2} \\ \mu_{A2} & \mu_{B2} \end{bmatrix} \begin{bmatrix} S_A \\ S_B \end{bmatrix} + \begin{bmatrix} \lambda_{02} \\ \mu_{02} \end{bmatrix},$$

where the coefficients are obtained from the following algebraic equations:

$$\rho\lambda_{A2} = \frac{a_{A2}(b_{A2}(1+b_{A2}c_A)+a_{B2}(2-2a_{A2}c_A+b_{B2}c_B))^2}{(b_{A2}b_{B2}-4a_{A2}a_{B2})^2} - \frac{1}{4}(4\delta_{A2}\lambda_{A2}+\beta_{A2}^2\lambda_{A2}^2+2\beta_{B2}^2\mu_{B2}\lambda_{B2}), \quad (73)$$

$$\rho\lambda_{B2} = \tfrac{1}{4}(-\beta_{A2}^2\lambda_{A2}^2-4\delta_{B2}\lambda_{B2}-2\beta_{B2}^2\mu_{B2}\lambda_{B2}), \quad (74)$$

$$\rho\mu_{A2} = \tfrac{1}{4}(-\beta_{B2}^2\mu_{B2}^2-4\delta_{A2}\mu_{A2}-2\beta_{A2}^2\mu_{A2}\lambda_{A2}), \quad \text{and} \quad (75)$$

$$\rho\mu_{B2} = \frac{a_{B2}(b_{B2}(1+b_{B2}c_B)+a_{A2}(2-2a_{B2}c_B+b_{B2}c_A))^2}{(b_{A2}b_{B2}-4a_{A2}a_{B2})^2} - \frac{1}{4}(4\delta_{B2}\mu_{B2}+\beta_{B2}^2\mu_{B2}^2+2\beta_{A2}^2\lambda_{A2}\mu_{A2}). \quad (76)$$

Thus, the postcrisis feedback advertising strategies are

$$u_{A2}^*(S_A(t), S_B(t)) = (M(t)-S_A(t)-S_B(t))(0.5\beta_{A2}\lambda_{A2})^2 \quad \text{and}$$
$$u_{B2}^*(S_A(t), S_B(t)) = (M(t)-S_A(t)-S_B(t))(0.5\beta_{B2}\mu_{B2})^2.$$

In the precrisis regime, the HJB equations are

$$(\rho+\chi)V_i(S_A, S_B) = \max_{p_{i1}, u_{i1}} \left\{ (p_{i1}-c_i)(1-a_{i1}p_{i1}+b_{i1}p_{3-i,1})S_i - u_{i1} + \chi W_i((1-\phi_A)S_A, (1-\phi_B)S_B) + \sum_i \frac{\partial V_i}{\partial S_i}\left(\beta_{i1}\sqrt{u_{i1}}\sqrt{M-S_A-S_B}-\delta_{i1}S_i\right)\right\}. \quad (77)$$

By differentiating (77) w.r.t. u_{i1} and p_{i1}, we obtain the FOCs:

$$-1+\frac{\beta_{i1}\sqrt{M-S_A-S_B}}{2\sqrt{u_{i1}}}\partial V_i/\partial S_i = 0$$

$$\Rightarrow \quad u_{i1}^*(S_A, S_B) = (M-S_A-S_B)(0.5\beta_{i1}\partial V_i/\partial S_i)^2, \quad (78)$$

and $p_{i1}=(1+a_{i1}c_i+b_{i1}p_{3-i,1})/2a_{i1}$, which yields the optimal prices:

$$p_{A1}^* = \frac{b_{A1}+a_{B1}(2+2a_{A1}c_A+b_{A1}c_B)}{4a_{A1}a_{B1}-b_{A1}b_{B1}} \quad \text{and} \quad p_{B1}^* = \frac{b_{A1}+a_{A1}(2+2a_{B1}c_B+b_{B1}c_A)}{4a_{A1}a_{B1}-b_{A1}b_{B1}}. \quad (79)$$

We substitute (78) and (79) in (77) to obtain the system of ODEs in $V_i(S_A, S_B)$ whose solutions are given by

$$\begin{bmatrix} V_A(S_A, S_B) \\ V_B(S_A, S_B) \end{bmatrix} = \begin{bmatrix} \lambda_{A1} & \lambda_{B1} \\ \mu_{A1} & \mu_{B1} \end{bmatrix} \begin{bmatrix} S_A \\ S_B \end{bmatrix} + \begin{bmatrix} \lambda_{01} \\ \mu_{01} \end{bmatrix},$$

where the coefficients are obtained from the following algebraic equations:

$$(\rho+\chi)\lambda_{A1} = \frac{a_{A1}(b_{A1}(1+b_{A1}c_A)+a_{B1}(2-2a_{A1}c_A+b_{A1}c_B))^2}{(b_{A1}b_{B1}-4a_{A1}a_{B1})^2} - \frac{4\delta_{A1}\lambda_{A1}+\beta_{A1}^2\lambda_{A1}^2+2\beta_{B1}^2\mu_{B1}\lambda_{B1}-4\chi(1-\phi_A)\lambda_{A2}}{4}, \quad (80)$$

$$(\rho+\chi)\lambda_{B1} = \tfrac{1}{4}(-\beta_{A1}^2\lambda_{A1}^2-4\delta_{B1}\lambda_{B1}-2\beta_{B1}^2\mu_{B1}\lambda_{B1}+4\chi(1-\phi_B)\lambda_{B2}), \quad (81)$$

$$(\rho+\chi)\mu_{A1} = \tfrac{1}{4}(-\beta_{B1}^2\mu_{B1}^2-4\delta_{A1}\mu_{A1}-2\beta_{A1}^2\mu_{A1}\lambda_{A1}+4\chi(1-\phi_A)\mu_{A2}), \quad (82)$$

and

$$(\rho+\chi)\mu_{B1} = \frac{a_{B1}(b_{B1}(1+b_{B1}c_B)+a_{A1}(2-2a_{A1}c_B+b_{B1}c_A))^2}{(b_{A1}b_{B1}-4a_{A1}a_{B1})^2} - \frac{4\delta_{B1}\mu_{B1}+\beta_{B1}^2\mu_{B1}^2+2\beta_{A1}^2\lambda_{A1}\mu_{A1}-4\chi(1-\phi_B)\mu_{B2}}{4}. \quad (83)$$

Thus, the precrisis feedback advertising strategies are

$$u_{A1}^*(S_A(t), S_B(t)) = (M(t)-S_A(t)-S_B(t))(0.5\beta_{A1}\lambda_{A1})^2 \quad \text{and}$$
$$u_{B1}^*(S_A(t), S_B(t)) = (M(t)-S_A(t)-S_B(t))(0.5\beta_{B1}\mu_{B1})^2. \quad \square$$

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